United States
Department of
Agriculture

Forest Service

Southern
Research Station

Research Note
SRS-10

November 2001

Remarks on Height-Diameter Modeling

Lei Yuancai and Bernard R. Parresol[1]

Abstract

Height-diameter model forms in earlier published papers are examined. The selection criteria used in height-diameter model forms are not reasonable when considering tree biological growth pattern. During model selection, forms for height-diameter relationships should include consideration of both data-related and reasonable biological criteria, not just data-related criteria. A reasonable model form should possess the S-shaped functional properties of monotonic increment ($dy/dt > 0$), inflection point ($d^2y/dt^2 = 0$), and asymptotical value ($y \rightarrow$asymptote as $t \rightarrow \infty$), rather than the concave shaped, functional properties. The S-shaped models are best when appropriate data are collected (data sets containing early growth). The Bertalanffy-Richards and Schnute functions are recommended for modeling height-diameter relationships, and an example contrasts the fit of the sigmoidal Schnute function against the concave Meyer function.

Keywords: Height-diameter, model form, Schnute function, selection criteria, sigmoid growth.

Introduction

Foresters often use height-diameter models to predict total tree height (*H*) based on observed diameter at breast height (d.b.h.) for estimating tree or stand volume and site quality, e.g., Stout and Shumway 1982. Therefore, estimations of tree or stand volume and site quality rely heavily on accurate height-diameter functions. There have been many postulated height-diameter models for different species and regions since Meyer (1940) suggested the exponential height-diameter relationship:

$$H = k + b_1(1 - e^{-b_2 DBH})$$

[1] Staff Member, Research Institute of Forest Resource Information Techniques, Chinese Academy of Forestry, Beijing 100091, People's Republic of China; Mathematical Statistician, U.S. Department of Agriculture, Forest Service, Southern Research Station, Southern Forest Inventory, Monitoring, and Analysis Program, Asheville, NC 28804, respectively.

where

k = breast height {4.5 feet (ft) [1.37 meters (m)] aboveground level in the United States, 1.3 m (4.3 ft) in countries using the metric system}; and
b_i = a coefficient.

Sigmoid and Concave Curves: Competing Functions

Huang and others (1992) selected and compared 20 nonlinear height-diameter functions, fitted by weighted, nonlinear least-squares regression for white spruce (*Picea glauca* Voss.) and aspen (*Populus tremuloides* Michx.). Their selection was based on examination of height-diameter relationships as revealed by plotting *H* against *DBH* for the two species. Based on their examinations, Huang and others (1992) believed that the height-diameter relationship for white spruce was a typical sigmoid or S-shaped curve with an inflection point occurring in the lower portion of the data points (fig. 1). The height-diameter relationship for aspen may be considered a concave curve with no apparent inflection point (fig. 2). Following their graphical examinations, the authors proposed that functions generating concave shapes and functions generating sigmoid shapes should be considered in the process of functional selection in order to get an accurate height-diameter model. Thus, they selected and compared 20 published, nonlinear height-diameter functions for 16 tree species in Alberta, Canada. Interestingly, based on the *t*-statistics for significance of parameters, mean squared error (MSE) values, and the plot of studentized residuals against the predicted heights, all the equations recommended for height-diameter relationships from their study produced S-shaped curves with inflection points.

Fang and Bailey (1998) investigated 33 height-diameter equations, including S-shaped and concave-shaped curves, for tropical forests on Hainan Island in southern China. They assumed that height-diameter relationships could be described by height-growth curves with age replaced by

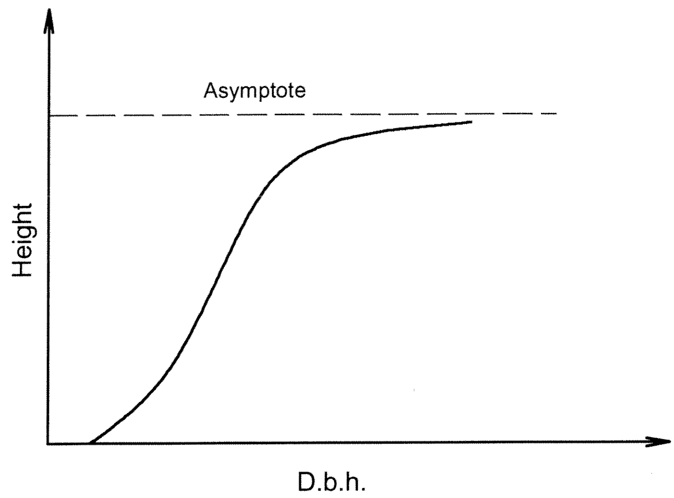

Figure 1—Sigmoidal curve.

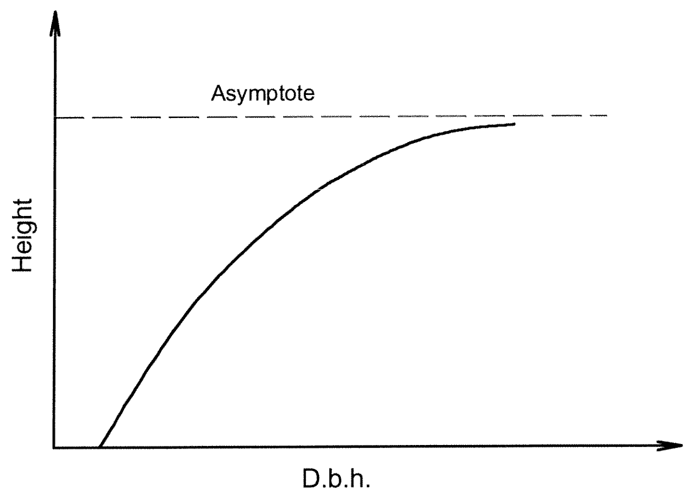

Figure 2—Concave curve.

d.b.h. They considered a number of characteristics in selecting functions but neglected functional inflection point as a selection criterion. Based on MSE values, most of the best equations with three parameters gave S-shaped curves with inflection points.

Biological Growth Pattern Points to Desirable Mathematical Properties

In addition to Huang and others (1992) and Fang and Bailey (1998), many others have used both S-shaped and concave-shaped curves in forest models for height-diameter curves, e.g., Arabatzis and Burkhart 1992. However it seems that in the course of model selection, none of these published papers considered the biological process of tree growth. From a biological point of view, a curve of height growth does exhibit a sigmoid or S-shaped pattern for a normal tree life span (Chaplain and others 1999, Clutter and others 1983, Thompson and others 1992). From a mathematical perspective, curves for height growth should possess the properties of (1) monotonic increment, (2) inflection point, and (3) asymptotic value. These properties can be obtained by taking the first derivative of the dependent variable (y) with respect to the independent variable (t) and setting it to be greater than zero, i.e., $dy/dt > 0$, for monotonic increment, by taking the second derivative to be equal to zero for inflection point, i.e., $d^2y/dt^2 = 0$, or by constraining the dependent variable to approach an asymptote as the independent variable goes to infinity, i.e., $y \rightarrow$ asymptote as $t \rightarrow \infty$.

Ambiguity

Why were both the S-shaped and the concave-shaped functional forms always considered as competing models in the selection process? We think that some characteristics of such functions for height-diameter relationships and the biological process of tree growth are understood ambiguously, so that comparisons of functional forms often have been based on empirical criteria, i.e., data-related criteria, with little regard to the characteristics imposed *a priori* by the functional forms on the estimated curve. It is important to have evaluated and acquired an understanding of the assumptions and implicit limitations imposed by these commonly used models. This is because the inherent mathematical characteristics of a given functional form have important implications (1) for the hypotheses that can be tested, (2) for the statistical estimation methods employed, and (3) on the shape of the resulting height-diameter curve. Also, considerable bias can arise due to the use of an inappropriate functional form.

Therefore, at least three important characteristics of height-diameter curves should be examined before fitting a model: (1) monotonic increment, (2) inflection point, and (3) asymptote, e.g., Parresol 1992, p. 1,430. An S-shaped curve has the properties of monotonic increment, inflection point, and asymptotic value, while a concave-shaped curve has properties of monotonic increment and asymptotic value, but has no inflection point. Why was an S-shaped model better than a concave-shaped one in the reviewed papers? This is because those characteristics exhibited by the S-shaped models are more appropriate to describe a realistic height-diameter relationship that exhibits an S-shaped biological growth pattern.

Previous height-diameter models may have been concave because only trees larger than the inflection point were in the database. In this case, the concave model may have better fit statistics than an S-shaped model, but it may be poor in the lower range where there are no data (poor extrapolation properties). Thus, the S-shaped model reflects appropriate biological properties that are not captured by the data-driven concave model but may exhibit slightly poorer fit statistics. This may be why some researchers have felt that the concave model was superior. Today, however, when total biomass utilization and global carbon-uptake are predominant issues, we also are interested in smaller trees. Thus, the S-shaped models should be considered more appropriate than ever before.

Recommendations

In light of our reviews and discussions, we propose and recommend the following for modeling height-diameter relationships:

1. The selection of a functional form for height-diameter relationships should not be restricted to the ease-of-fit to data, nor only to data-related criteria, but also should consider characteristics of the chosen model, such as monotonic increment, functional inflection point, and asymptotic value. These characteristics are very important (especially for making extrapolations) and easy to test for the chosen model forms—to see whether or not they are suitable for modeling height-diameter relationships. A functional form with monotonic increment, inflection point, and asymptotic value can describe height-diameter relationships appropriately. The curve of the functional form should be typical of a height cumulative growth curve, which starts at the origin value, increases steadily to attain maximum growth at an inflection point, and then gradually approaches an asymptotical value. This type of curve is also directly compatible with a height-increment curve, where the increment begins at a value of zero, increases steadily to reach a maximum, and then decreases asymptotically toward zero.

2. Data for model building should include early height growth. Functions generating a concave-shaped curve as competing models to depict height over diameter should not be necessary. Generally, functions rendering a concave curve cannot describe tree or stand growth behaviors appropriately (especially fast-growth trees and stands). However, if a data set includes only larger or older trees beyond the inflection point, then a model generating a concave curve will probably work best. Even though a sigmoid model is correct theoretically, it may not give a good fit if the range of the data is beyond the point of inflection for the species and growing conditions.

3. Sigmoid or S-shaped functions may be expanded to model other forestry relationships—such as volume-age, height-age, and diameter-age functions.

4. The Bertalanffy-Richards growth function (Richards 1959) and the Schnute (1981) models are probably the most flexible and versatile functions available for modeling height-diameter relationships. Both are able to assume various shapes with different parameter values, and they produce satisfactory curves under wide-ranging biological/ecological modeling circumstances. The Schnute model is easy to fit and quick to achieve convergence for any database (Lei 1998). The Bertalanffy-Richards function is

$$H = \begin{cases} b_1(1 - b_2 e^{-b_3 DBH})^{[1/(1-b_4)]}, \\ \quad \text{if } b_4 > 1 \text{ or } 0 \le b_4 < 1 \text{ or instances when } b_4 < 0 \\ \\ b_1(b_2 e^{b_3 DBH} - 1)^{[1/(1-b_4)]}, \\ \quad \text{under certain conditions when } b_4 < 0 \end{cases}$$

The Schnute function is

$$H = \left[H_1^b + (H_2^b - H_1^b)\frac{1 - e^{-a(DBH - DBH_1)}}{1 - e^{-a(DBH_2 - DBH_1)}} \right]^{(1/b)}$$

where

DBH_1 = diameter of a young tree (lower range of data),
DBH_2 = diameter of an old tree (upper range of data),
H_1 = tree height at DBH_1,
H_2 = tree height at DBH_2,
a = constant acceleration in growth rate, and
b = incremental acceleration in growth rate.

The coefficients to be estimated in the Schnute function are H_1, H_2, a, and b. The values DBH_1 and DBH_2 are fixed and are normally taken to be the smallest and largest diameters in the data.

Example

Consider the 40 height-diameter pairs listed in table 1. These data are for *Homalium racemosum* Jacq. growing in the Caribbean National Forest on the tropical island of Puerto Rico and represent a typical species growing in the subtropical wet life zone (Holdridge 1967). These data points are graphed in figure 3 along with the fitted sigmoidal Schnute function

Table 1—Height-diameter pairs for 40 measured trees of the species *Homalium racemosum* from the Caribbean National Forest on the island of Puerto Rico

Tree	D.b.h.	Height	Tree	D.b.h.	Height
	Cm	*m*		*Cm*	*m*
1	12.7	15.2	21	29.0	21.3
2	14.2	10.7	22	29.5	18.3
3	15.5	15.2	23	29.7	16.8
4	15.7	16.8	24	32.3	19.8
5	15.7	16.8	25	32.8	18.3
6	18.5	16.8	26	33.5	19.8
7	18.5	18.3	27	35.1	24.4
8	18.8	13.7	28	36.6	19.8
9	19.6	12.2	29	36.6	21.3
10	20.3	19.8	30	37.3	21.3
11	20.8	16.8	31	37.6	21.3
12	21.6	15.2	32	40.6	21.3
13	21.8	15.2	33	41.7	24.4
14	22.4	16.8	34	41.9	22.9
15	23.4	21.3	35	48.8	24.4
16	25.1	18.3	36	53.8	22.9
17	25.7	15.2	37	55.9	19.8
18	28.2	21.3	38	56.1	24.4
19	28.4	18.3	39	77.2	25.9
20	28.7	13.7	40	81.5	25.9

D.b.h. = diameter at breast height.

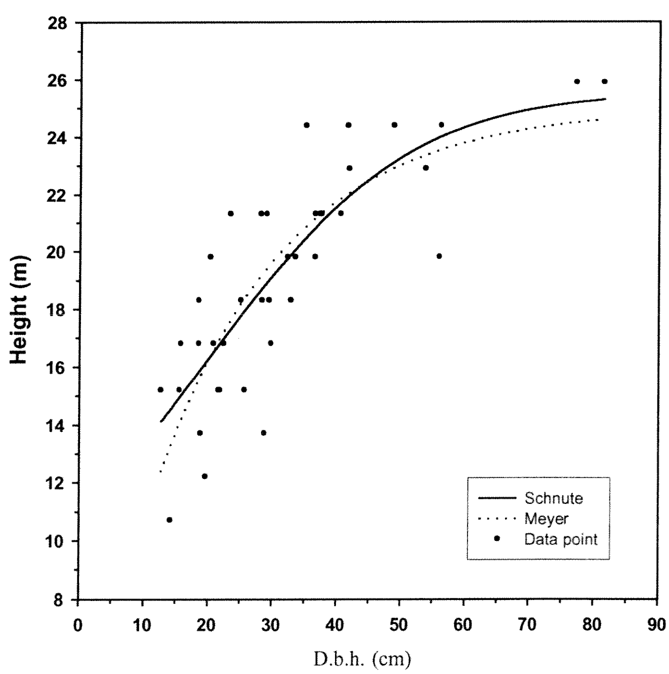

Figure 3—Height-diameter scatter plot of 40 trees of *Homalium racemosum* Jacq. from Puerto Rico showing a sigmoid curve (Schnute function) and a concave curve (Meyer function) through the data points.

$$H = \left[14.1084^{-2.7486} + (25.2927^{-2.7486} - 14.1084^{-2.7486}) \frac{1 - e^{-0.0709(DBH-12.7)}}{1 - e^{-0.0709(81.5-12.7)}} \right]^{\left(\frac{1}{-2.7486}\right)}$$

$$R^2 = 0.67,\ RMSE = 2.26 \text{ m}$$

and the fitted concave Meyer function

$$H = 1.3 + 23.7065(1 - e^{-0.0497\,DBH})$$

$$R^2 = 0.65,\ RMSE = 2.28 \text{ m}$$

where

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(H_i - \hat{H}_i)^2}{\sum_{i=1}^{n}(H_i - \bar{H})^2}$$

$$RMSE = \text{root mean squared error} = \sqrt{\sum_{i=1}^{n}(H_i - \hat{H}_i)^2 / (n-p)}\ ,$$

n = number of observations, and
p = number of function parameters.

It is obvious from figure 3 that the sigmoidal curve is better suited to the data than the concave curve. The asymptote (A) and the inflection point (DBH^*, H^*) for the Schnute function are:

$$A = \left[\frac{e^{a \cdot DBH_2} H_2^b - e^{a \cdot DBH_1} H_1^b}{e^{a \cdot DBH_2} - e^{a \cdot DBH_1}} \right]^{(1/b)} = 25.6 \text{ m}$$

$$DBH^* = DBH_1 + DBH_2 - \frac{1}{a} \ln \left[\frac{b(e^{a \cdot DBH_2} H_2^b - e^{a \cdot DBH_1} H_1^b)}{H_2^b - H_1^b} \right] = 18.4 \text{ cm}$$

$$H^* = \left[\frac{(1-b)(e^{a \cdot DBH_2} H_2^b - e^{a \cdot DBH_1} H_1^b)}{e^{a \cdot DBH_2} - e^{a \cdot DBH_1}} \right]^{(1/b)} = 15.8 \text{ m}$$

In summary, data-related and reasonable biological criteria should be considered simultaneously for estimating forestry relationships in the process of selecting functions.

Acknowledgments

We are grateful to Drs. Pacheco Marques and Boris Zeide for their many helpful suggestions. Thanks also to Drs. Clark Baldwin and Stan Zarnoch for their reviews.

Literature Cited

Arabatzis, A.A.; Burkhart, H.E. 1992. An evaluation of sampling methods and model forms for estimating height-diameter relationships in loblolly pine plantations. Forest Science. 38 (1): 192–198.

Chaplain, M.A.J.; Singh, G.D.; McLachlan, J.C., eds. 1999. On growth and form: spatio-temporal pattern formation in biology. New York: John Wiley and Sons. 436 p.

Clutter, J.L.; Fortson, J.C.; Pienaar, L.V. [and others]. 1983. Timber management: a quantitative approach. New York: John Wiley and Sons. 333 p.

Fang, Z.; Bailey, R.L. 1998. Height-diameter models for tropical forest on Hainan Island in southern China. Forest Ecology and Management. 110: 315–327.

Holdridge, L.R. 1967. Life zone ecology. San Jose, Costa Rica: Tropical Science Center. 206 p.

Huang, S.; Titus, S.J.; Wiens, D.P. 1992. Comparison of nonlinear height-diameter functions for major Alberta tree species. Canadian Journal of Forest Research. 22: 1297–1304.

Lei, Y. 1998. Modelling forest growth and yield of *Eucalyptus globulus* Labill. in central-interior Portugal. Vila Real, Portugal: Universidade de Trás-os-Montes e Alto Douro. 155 p. Ph.D. dissertation.

Meyer, H.A. 1940. A mathematical expression for height curves. Journal of Forestry. 38: 415–420.

Parresol, B.R. 1992. Baldcypress height-diameter equations and their prediction confidence intervals. Canadian Journal of Forest Research. 22: 1429–1434.

Richards, J. 1959. A flexible growth function for empirical use. Journal of Experimental Botany. 10 (29): 290–300.

Schnute, J. 1981. A versatile growth model with statistically stable parameters. Canadian Journal of Fisheries and Aquatics Science. 38: 1128–1140.

Stout, B.B.; Shumway, D.L. 1982. Site quality estimation using height and diameter. Forest Science. 28: 639–645.

Thompson, D'A.; Bonner, J.T.; Gould, S.J. 1992. On growth and form. Cambridge, UK: Cambridge University Press. 368 p.

☆U.S. GOVERNMENT PRINTING OFFICE: 2001-765-022/00058

Southern Research Station, U.S. Department of Agriculture, Forest Service
P.O. Box 2680, Asheville, NC 28802