

Simple, flexible, trigonometric taper equations

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Received July 12, 1990

Accepted February 4, 1991

THOMAS, C. E., and PARRESOL, B. R. 1991. Simple, flexible, trigonometric taper equations. *Can. J. For. Res.* **21**: 1132-1137.

There have been numerous approaches to modeling stem form in recent decades. The majority have concentrated on the simpler coniferous bole form and have become increasingly complex mathematical expressions. Use of trigonometric equations provides a simple expression of taper that is flexible enough to fit both coniferous and hardwood bole forms. As an illustration, we applied trigonometric taper equations to examples from thinned and unthinned slash pine (*Pinus elliottii* Engelm. var. *elliottii*), willow oak (*Quercus phellos* L.), and sweet gum (*Liquidambar styraciflua* L.). Comparison of new trigonometric models with a segmented-polynomial approach developed in 1976 indicates that equations based on trigonometric functions perform equally well and have real advantages in terms of parsimony.

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Il y a eu de nombreuses approches pour modéliser la forme de la tige au cours des dernières décennies. La plupart d'entre elles, bien que se limitant à la forme plutôt simple du tronc des résineux, se traduisent par des équations mathématiques de plus en plus complexes. L'utilisation d'équations trigonométriques fournit une expression simple du défilement tout en étant assez flexible pour s'ajuster à la forme du tronc des résineux et des feuillus. À titre d'exemple, nous avons appliqué des équations trigonométriques de défilement à des tiges de pin de Floride (*Pinus elliottii* Engelm. var. *elliottii*), de chêne Saule (*Quercus phellos* L.) et de liquidambar d'Amérique (*Liquidambar styraciflua* L.) provenant de peuplements éclaircis et non éclaircis. La comparaison des nouveaux modèles trigonométriques avec une approche procédant par segments polinômiaux développée en 1976 indique que les équations basées sur des fonctions trigonométriques conviennent aussi bien et possèdent des avantages notables quant à la parcimonie.

Introduction

Modeling stem form mathematically has been a wide-ranging effort in forestry. Approaches have included simple hyperbolic expressions (Behre 1924, 1927), multiple regression on high-order polynomials (Bruce et al. 1968), complicated multiexponent power equations (Demaerschalk 1973; Demaerschalk and Kozak 1977), and segmented-polynomial equations conditioned to join smoothly (Max and Burkhart 1976). Even some simple sine function transformations for stem form were reported by Bitterlich (1976) for use with the Relascop. Reviews of the many other taper equations are included in several taper-related papers (e.g., Cao et al. 1980).

The purpose of this study was to examine the applicability of trigonometric function-based taper equations and to compare them with a widely recognized and flexible taper equation of well-known reliability. To demonstrate the flexibility of the general model, we selected data that included both simple softwood bole form and rather more complicated hardwood bole form.

Materials

Taper data for slash pine (*Pinus elliottii* Engelm. var. *elliottii*), willow oak (*Quercus phellos* L.), and sweet gum (*Liquidambar styraciflua* L.) were available to examine the performance of the trigonometric taper equation and to compare its performance with that of the segmented-polynomial equation of Max and Burkhart (1976). The slash pine data are described by Lohrey (1984). Briefly, they consist of 199 felled plantation-grown pine trees, 107 of which were from unthinned stands and 92 from stands that had been thinned from below. All trees were cut at a 15-cm stump height. Diameter outside bark and diameter inside bark were measured at 0.15, 0.6, and 1.3 m and every 1.5 m along the remainder of the stem. Total tree height was also measured. Section volumes inside bark were calculated using Smalian's formula. Table 1 gives some simple descriptive statistics on the two data sets (thinned and unthinned).

For willow oak we obtained data from Bryce Schlaegel, USDA Forest Service R-8 Timber Management. Data collection and statistics are reported in Schlaegel (1981); we briefly review his descriptions here. Data were collected from 10 natural bottomland hardwood stands in Mississippi. Trees were chosen for destructive sampling from uneven-aged mixed-species stands. Measurements were made at regular intervals along the bole. Table 2 presents some simple descriptive statistics for these data. A similar description of data collection and statistics for sweet gum is reported in Schlaegel (1984). Statistics for the sweet gum data are given in our Table 2.

Methods

Trigonometric functions on the unit circle have a direct analogy to the relative height - relative diameter plots presented in many taper equations. They are limited in the values that may be assumed to the range of -1 to 1 for sines and cosines and 0 to $+\infty$ for tangents and cotangents. Trigonometric functions have also been expressed as Taylor series of high-order polynomials. In fact, tabulated values for complex trigonometric functions were produced by computing the values using the polynomial formulae. These characteristics suggest that trigonometric functions might provide a compact and simple method for expressing bole taper. For example, the Taylor series expression for a sine function centered at $x = 0$ is

$$\sin x = (x/1!) - (x^3/3!) + (x^5/5!) + \dots \\ + [(-1)^{n+1}x^{2n+1}/(2n+1)!] \quad n = 1, 2, 3, \dots$$

TABLE 1. Characteristics of slash pine plantation trees

	Unthinned stands	Thinned stands
No. of trees	107	92
Mean dbh (cm)	16.8	25.4
dbh range (cm)	5.6-33.0	6.6-48.3
Mean height (m)	16.2	20.2
Height range (m)	6.1-21.7	8.8-33.5

TABLE 2. Characteristics of willow oak and sweet gum trees

	Willow oak	Sweet gum
No. of trees	62	45
Mean dbh (cm)	48.9	41.2
dbh range (cm)	17.3-94.0	15.2-82.5
Mean height (m)	26.9	27.6
Height range (m)	17.4-37.8	14.6-39.0

After some examination of the plots of unit-circle trigonometric functions, including nonzero centered transformations (of the form $\sin(x + a\pi)$) and comparison to the plots of tree taper¹ on a relative height and diameter scale, we selected the following taper model:

$$[1] \quad d^2/D^2 = b_1(x - 1) + b_2 \sin(c\pi x) + b_3 \cotan(\pi x/2)$$

where d is diameter at a given height, D is diameter at breast height (the left-hand side of eq. 1 will also be referred to as relative square diameter), x is the corresponding relative height, i.e., height of observation/total height above ground, and c is a coefficient. Arguments for trigonometric functions are expressed in radians. The cotangent function yields values from $+\infty$ to 0 when x is scaled from 0 to $\pi/2$. The initial selection of expressions of $c\pi$ (for arguments of the sine function) were based on the interval necessary to condition the sine function to assume values from -1 to 1 . While a model that uses $c = 2$ performs quite well for the hardwoods, bole form did not agree as well for the slash pine. Therefore, we ran a series of nonlinear regressions to determine an appropriate value for c . Values of c for the slash pine ranged from 1.4 to 1.5, and for the hardwoods c ranged from 1.9 to 2.2. It is possible that nonlinear fitting for this additional parameter would be reasonable. However, we found little improvement in R^2 or mean squared error over using fixed values of 1.5 for the softwood and 2.0 for the hardwood.

The proliferation of taper models over the last 2 decades provides a wide variety of models that could be used to compare performance. Results of several applications for growth and yield models suggested that the model of Max and Burkhart (1976) for southern tree species was well behaved and well tested. Their model was

$$[2] \quad d^2/D^2 = (b_1(x - 1) + b_2(x^2 - 1) \\ + b_3(a_1 - x)^2 I_1 + b_4(a_2 - x)^2 I_2)$$

where

$$I_1 = 1 \text{ if } x < a_1, \text{ otherwise } I_1 = 0 \\ I_2 = 1 \text{ if } x < a_2, \text{ otherwise } I_2 = 0$$

¹As with most model fitting, this was actually an iterative procedure, involving examination of raw data and transformed data, preliminary fitting of the data, and then iteratively examining plots of residuals from a variety of trigonometric models.

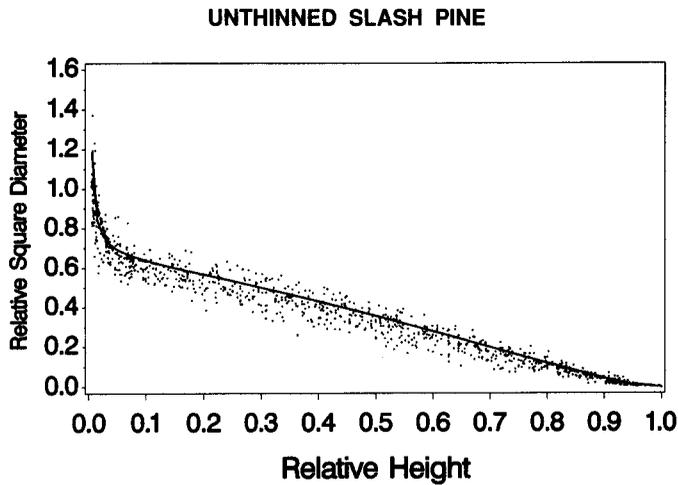


FIG. 1. Plot of relative height versus relative square diameter inside bark for unthinned slash pine. Solid line represents the trigonometric function based equation; broken line represents the Max and Burkhart segmented-polynomial equation.

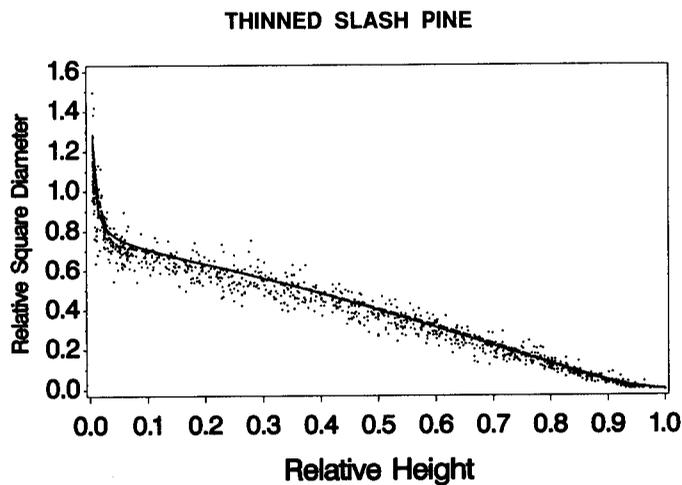


FIG. 2. Plot of relative height versus relative square diameter inside bark for thinned slash pine. Solid line represents the trigonometric function based equation; broken line represents the Max and Burkhart segmented-polynomial equation.

and other variables are as previously defined (a_i and b_i are coefficients). This provides for a three-segment polynomial, characterized by quadratic degree for lower bole, central bole, and top.

Results and discussion

The results will be discussed in two sections: first, the simpler bole form of slash pine and then the more complex bole form of willow oak. We will also present results for sweet gum, but little discussion will be spent on it, because it has a form very similar to that of willow oak and because it neither required special processing or coefficients, nor produced any unusual or interesting conditions different from the other two species.

Slash pine

We transformed the raw data to obtain relative height and square diameter and to fit the trigonometric model using ordinary least squares regression. As we noted earlier, nonlinear models were run to test the value of using a

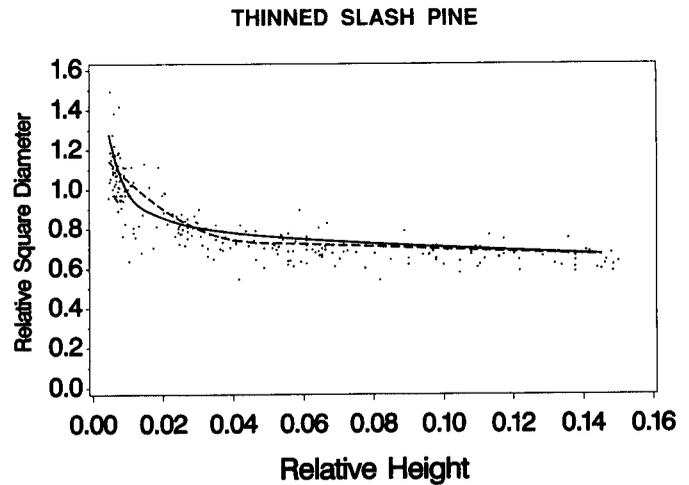


FIG. 3. Plot of lower portion of bole, depicting differences between the two model fits in this portion of the thinned slash pine bole. Solid line represents the trigonometric function based equation; broken line represents the Max and Burkhart segmented-polynomial equation.

variable coefficient (c) for the argument of the sine function. This resulted in the selection of $c = 1.5$. We examined the models from fixed and variable coefficients for the argument and found little difference between the two. Therefore, we have chosen to report results from the simpler model, which does not involve a variable coefficient for the sine function and the concomitant nonlinear model fitting.

In contrast, it was necessary to use nonlinear regression to determine the join point parameters in the Max and Burkhart model. Figure 1 illustrates the data and computed inside-bark taper equation lines for unthinned slash pine for each model. The solid line represents the trigonometric function, and the broken line represents Max and Burkhart's model. Similarly, Fig. 2 represents the inside-bark models for thinned slash pine. Finally, a blowup of the lower portion of the bole of thinned stems is presented in Fig. 3; a similar result was obtained for unthinned slash pine. Note that for these figures we randomly eliminated 25% of the observations to more clearly display the differences between the two model lines.

After fitting the equations we plotted the residuals and compared mean squared errors measured in the original values for both the models. Statistics for the two models are presented in Table 3. The fit index is a corrected coefficient of determination, the result of determining residual values from the data as measured in the original units.

Examination of both the graphical and statistical presentations provides evidence for the similarities and minor differences between the two models. Both models differentiated between the thinned and unthinned stem forms. Confidence intervals about parameter values for the thinned and unthinned trigonometric models did not overlap. The asymptotic confidence intervals for the nonlinear models generated by SAS Institute Inc. (1985) software product PROC NLIN overlapped significantly. We have some serious reservations about these nonlinear confidence intervals. However, their validity was not germane to the trigonometric model. The trigonometric model is more parsimonious (has fewer parameters to estimate) and requires less computation in fitting the model. The usual statistics for the two models provide little to choose between them.

TABLE 3. Results from the ANOVA for regression models describing the inside-bark bole form of thinned and unthinned slash pine

Model	Coefficient values						Statistics	
	b_1	b_2	b_3	b_4	a_1	a_2	SE*	FI [†]
Unthinned								
Trigonometric	-0.662	0.0278	0.004 56	—	—	—	0.305	0.983
Max-Burkhart	-4.154	1.988	-2.140	146.5	0.829	0.0545	0.291	0.984
Thinned								
Trigonometric	-0.741	0.0345	0.003 91	—	—	—	0.444	0.987
Max-Burkhart	-5.689	2.777	-3.028	223.3	0.843	0.0462	0.428	0.988

*Standard error of the estimate; expressed in the original units. Coefficients of variation about the mean volume (24 ft³ (0.68 m³)) were about 6%.

[†]Fit index: the expression of coefficient of determination in original units. $FI = 1 - [\Sigma(d - \bar{d})^2 / \Sigma(d - \bar{d})^2]$.

It is possible that butt flare in the trigonometric model could become extreme. This is because the cotangent goes to $+\infty$ at $x = 0$ (a transformation to the cotangent could be used to guarantee some maximum value at ground level). However, in our experience this concern was not warranted. Extreme flare occurred only in the stump portion of trees, not in the portion that would be utilized.² In sapling-sized trees, less than 5 cm in dbh, some strange shapes may result from application of the trigonometric model. However, the volume or biomass obtained from the equations are still reasonable estimates; therefore shape of these small stems should cause little practical concern.

Willow oak and sweet gum

Figure 4 illustrates the relative cross-sectional profile for the willow oak boles. Total relative height equals 1 for all-sized trees, and relative cross-sectional area is a proportion of basal area at breast height. Note that actual measurements end at stump height, not ground level. The slight bulge in the taper curve between 50 and 80% of relative height represents a portion of the stem (main bole) where taper slows at the base of the live crown and then resumes at a slightly more rapid rate within the deliquescent crown. In Fig. 4 the trigonometric model seems to provide a slightly better fit than the segmented-polynomial model. The statistical estimates for inside-bark taper equations from both trigonometric and Max and Burkhart's models are presented in Table 4.

As with the slash pine data, differences between the overall statistical fits obtained are minimal. The advantages remain in the simplicity and ease of application for the trigonometric model. While both models are conditioned to equal 0 at the tip of the tree, both equations may result in very small negative diameters near the tip. Practically, this should be of little concern. Biomass or gross volume still integrates to yield positive values, and the small amount of material should seldom yield unreasonable estimates of topwood.

Conclusions

Deriving suitable inside- (or outside-) bark taper equations can prove to be a valuable exercise. The equations can be

²While the taper function goes to infinity, integration of the taper function for volume or biomass does not involve an infinite value. Both volume and biomass integrals give finite solutions.

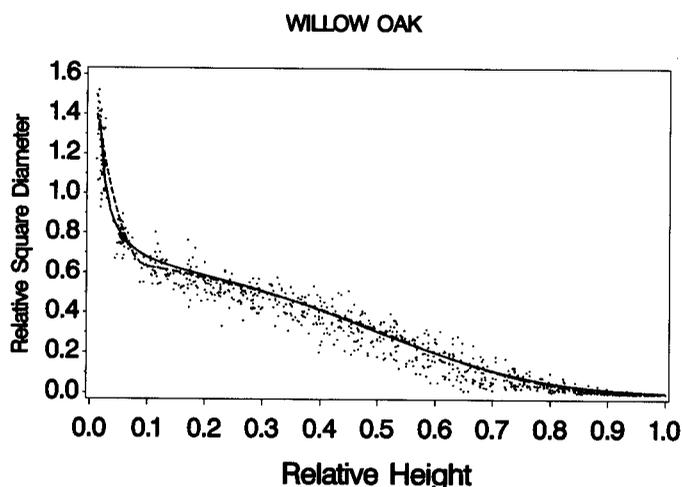


FIG. 4. Plot of relative height versus relative square diameter inside bark for willow oak. Solid line represents the trigonometric function based equation; broken line represents the Max and Burkhart segmented-polynomial equation.

used in determining product outturns from various stand conditions. Unthinned stands of slash pine were shown to have different taper equations than thinned stands. Trigonometric function based equations yield a single solution for any portion of the stem, while equations that have multiple quadratic forms require solution for volumes in parts of the tree bole that may require solving two equations. Solving several equations is not a problem once the taper equation is established; however, it could be a problem if new taper coefficients are needed, e.g., when new thinning densities are encountered, there would be six coefficients rather than three to fit. The sine-cosine functions have a range from -1 to 1 and tangent-cotangent functions, from $+\infty$ to 0 , when properly conditioned. They can be considered a parsimonious expression of high-order polynomials.

As a number of investigators have observed, integration of the taper equations yields a volume directly. Integration between intermediate heights or over log lengths provides volumes within sections of the bole. The form of the taper equation can make this process more or less complex. Integration of the quadratic forms for different logs within the bole means that more complex integrations may be necessary. The trigonometric formulation we have used is simple to integrate; each term in the equation can easily be solved.

TABLE 4. Results from the ANOVA for regression models for willow oak and sweet gum taper coefficients

Model	Coefficient values						Statistics	
	b_1	b_2	b_3	b_4	a_1	a_2	SE*	FI†
Willow oak								
Trigonometric	-0.569	0.0741	0.019 24	—	—	—	1.284	0.967
Max-Burkhart	-2.752	1.410	-2.200	96.3	0.553	0.0982	1.228	0.970
Sweet gum								
Trigonometric	-0.601	0.0588	0.0151	—	—	—	1.100	0.969
Max-Burkhart	-3.090	1.565	-1.719	127.2	0.677	0.0800	0.970	0.976

*Standard error of the estimate; expressed in the original units. Coefficients of variation about the mean volume (24 ft³ (0.68 m³)) were about 6%.

†Fit index: the expression of coefficient of determination in original units. $FI = 1 - [\Sigma(d - \hat{d})^2 / \Sigma(d - \bar{d})^2]$.

This provides another worthwhile advantage over joined quadratics.

To obtain volume, integrate over height

$$\hat{v} = k \int_{h_1}^{h_u} d^2 dh$$

where k is $\pi / 40\,000$, the factor for converting square centimetres to square metres; h_1 is the lower height limit; and h_u is the upper height limit. Performing a change of variable from h to x (relative height) results in

$$\hat{v} = kH \int_{x_1}^{x_u} d^2 dx$$

where H is total tree height, x_1 is h_1/H , and x_u is h_u/H . Specifically, integration yields

$$\hat{v} = kD^2H \int_{x_1}^{x_u} b_1x - b_1 + b_2\sin(c\pi x) + b_3\cot(\pi x/2) dx$$

$$\hat{v} = kD^2H \left\{ \frac{b_1}{2} (x_u^2 - x_1^2) - b_1(x_u - x_1) - \frac{b_2}{c\pi} [\cos(c\pi x_u) - \cos(c\pi x_1)] + \frac{2b_3}{\pi} \left[\ln \left(\frac{\sin(\pi x_u/2)}{\sin(\pi x_1/2)} \right) \right] \right\}$$

where \ln is the natural logarithm and $c = 1.5$ for slash pine and 2.0 for willow oak and sweet gum.

Integration of the segmented-polynomial equations is possible; however, the solutions are considerably more complex, always involving the integration of two equations in the butt log.

A final remark about inversion of the trigonometric form may be useful. The formulation we have solved is in terms of diameter inside (or outside) bark. It may occasionally be convenient to obtain a height to a known upper stem diameter. It is not obvious how to invert eq. 1 mathematically. It may be possible to invert the equation if the range of the sine and cotangent is limited. However, it is trivial to obtain the inverted solution using a simple iterative numerical technique, such as Newton-Raphson. Obtaining the derivative of the trigonometric equation is simple, which facilitates the numeric solution.

In previous publications, we have advocated a double-integral (density-volume) approach to determining weight or biomass of tree boles (e.g., Parresol and Thomas 1989). In those publications, we used simple taper equations to illustrate the method. This new taper equation should improve on the predictions when compared with predictions from prior taper models that we have used. We did not attempt to analyze differences between thinned and unthinned stands that might have resulted from tree size differences between the two conditions. This problem often occurs in thinning studies and is not easily resolved, and we feel, it is a topic worthy of treatment in its own right. In our study of bole forms for both hardwood and softwoods, trigonometric functions provide impressive flexibility, parsimony, and potential utility. For these reasons, we suggest that trigonometric functions be considered for developing and applying taper functions for many species.

Acknowledgement

We recognize the contribution of Dr. B. E. Schlaegel, who contributed hardwood data collected with great care during his years with the southern Hardwoods Project in Stoneville, Mississippi. He also contributed to early discussion of the taper project and had agreed to contribute to the writing and analysis. His untimely death was a loss to us as well as to the USDA Forest Service.

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