A simultaneous density-integral system for estimating stem profile and biomass: slash pine and willow oak

Bernard R. Parresol and Charles E. Thomas

Abstract: In the wood utilization industry, both stem profile and biomass are important quantities. The two have traditionally been estimated separately. The introduction of a density-integral method allows for coincident estimation of stem profile and biomass, based on the calculus of mass theory, and provides an alternative to weight-ratio methodology. In the initial development of the technique, sectional bole weight was predicted from a density integral formed from two equations that were fitted independently using ordinary least squares: (1) a stem-profile, or taper, function and (2) a specific gravity function. A test for contemporaneous correlations using slash pine (*Pinus elliottii* Engelm. var. *elliottii*) and willow oak (*Quercus phellos* L.) data showed highly significant correlations between the density integral and the stem-profile equation as well as the specific gravity equation. However, there was little or no correlation between the stem-profile and specific gravity equations. Because contemporaneous correlations exist between some of the equations, more efficient parameter estimation can be achieved through joint-generalized least squares, better known as seemingly unrelated regressions. However, the improvement in efficiency across parameters varies markedly based on the pattern of contemporaneous correlations. A simultaneous system of three equations was derived for slash pine and willow oak with nonlinear constraints across equations. Parameter estimates from seemingly unrelated regressions estimation had smaller standard errors in all cases than those from ordinary least squares estimation. For slash pine, standard errors were reduced by 11 to 29% and for willow oak, by 5 to 20%.

Résumé : À la fois le défilement de la tige et la biomasse revêtent une importance particulière pour l'industrie du bois. Ils ont traditionnellement été estimés séparément. Une méthode basée sur l'intégration de la masse et appelée méthode d'intégration de densité permet d'estimer simultanément le défilement et la biomasse tout en offrant une alternative à la méthode du ratio du poids. Au début du développement de la méthode, le poids des sections de tige était prédit par une intégrale de densité composée de deux équations ajustées séparément par la méthode classique des moindres carrés : (1) une équation du profil, ou du défilement, et (2) une équation du poids spécifique. Un test de corrélations simultanées sur les données du pin de Floride (*Pinus elliottii* Engelm. var. *elliottii*) et du chêne saule (*Quercus phellos* L.) montraient des corrélations fortement significatives entre l'intégrale de densité et l'équation du défilement ainsi que l'équation du poids spécifique. Cependant, il y avait peu ou pas de corrélation entre l'équation du défilement et celle du poids spécifique. Comme il existe des corrélations simultanées entre certaines de ces équations, on peut obtenir une meilleure estimation des paramètres par la méthode des moindres carrés unifiés et généraux, mieux connue sous le nom de méthode des régressions apparemment indépendantes. Toutefois, cette amélioration de l'efficacité au niveau des paramètres varie fortement selon le patron des corrélations simultanées. Un système de trois équations simultanées à contraintes non linéaires a été développé pour le pin de Floride et le chêne saule. Dans tous les cas, les paramètres estimés par la méthode des régressions apparemment indépendantes comportaient des erreurs standards plus faibles que celles qui étaient obtenues par la méthode classique des moindres carrés. Les réductions varient de 11 à 29% pour le pin de Floride et de 5 à 20% pour le chêne saule.

[Traduit par la Rédaction]

Introduction

As early as 1950, the use of weight as a measure of wood quantity has been practiced by many of the larger companies in North America and northern Europe (Husch et al. 1982). The use of weight measurement has steadily grown in importance with the increased demand for wood fiber and fuel. While weight has become increasingly important as a measure, so too stem form or taper has increased in importance to the sawnwood industry. The development of taper equations allows for prediction of product mix produced from a tree, and hence the economic value of trees (Busby and Ward 1989). Softwood species are particularly appropriate for the development of stem-profile, or taper, models because of the simplicity of stem form. Hardwood species are somewhat more difficult to model conceptually as well as practically because of the form of the stems in the crown and the tendency of the central stem to break up into smaller, but still significant branches. Use of weight in

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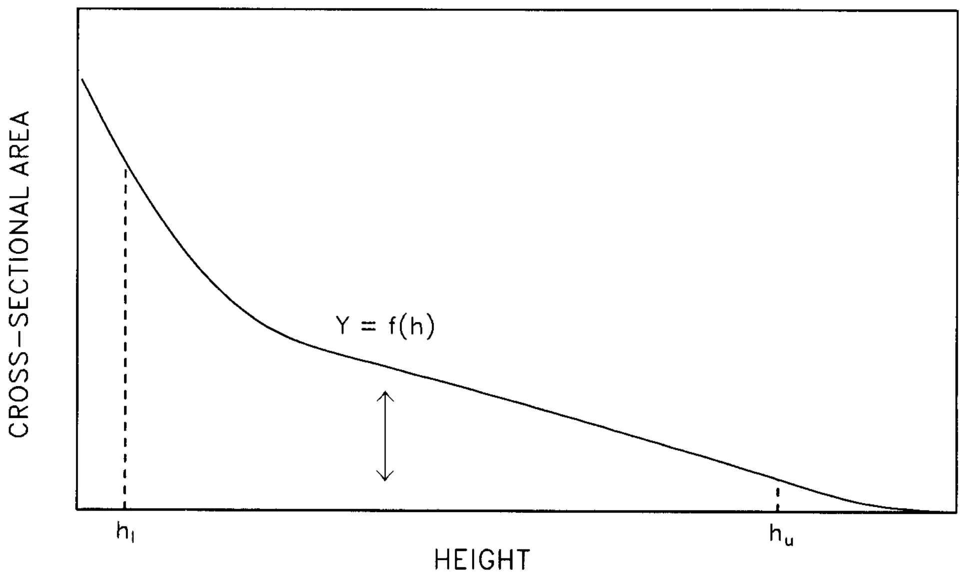
Fig. 1. Excurrent tree conceptualized as a two-dimensional lamina.



specifying the amount of hardwood is nonetheless appealing because of the recent increased interest in the utilization of smaller hardwood stems in chipboard and paper furnish.

This paper develops an integrated approach for stem biomass estimation that was first introduced by Parresol and Thomas (1989) using a piecewise approach. This new approach uses a generalized integral model for the prediction of dry-weight yield of wood for any portion of a tree bole, given diameter at breast height (DBH) and total tree height (H). This contrasts with the standard approach for predicting sectional bole weights, namely the use of ratio equations times total stem weight equations. We investigated the comparison of the weight ratio and the integrated approach in the earlier paper and found significant improvement in prediction using the integrated approach.

Methodology

The weight of an object is simply its volume multiplied by its density. It is the basis for calculation of bolt dry weights from sample trees used in biomass studies. The normal procedure is to cut disks from the base of each bolt for laboratory determination of its density or its specific gravity. Since the base of a bolt is the top of the previous bolt, a measure of density at the ends of each bolt (except the tip) is obtained. A weighted average bolt density (where weight is disk cross-sectional area) is determined, which is then multiplied by bolt volume to obtain bolt dry weight. The need for weighting density results from the fact that it varies throughout the tree. Specific gravity typically changes as a function of height within a tree stem, decreasing from base to tip (Husch et al. 1982; Parresol and Thomas 1987, 1989). Other factors such as age and size (i.e., DBH and H) may affect tree specific gravity (Taras and Wahlgren 1963).

Weight-ratio approach

When researchers first started fitting tree volume equations for different merchantability limits, they found, contrary to logic and fact, that the regression lines sometimes crossed each other. To circumvent this problem researchers devised a two-step method to calculate merchantable volume to any utilization specification. First, they estimated an equation to predict total tree volume. Second, they estimated an equation to predict the proportion or ratio of merchantable volume to total volume given the merchantability limits. When attention shifted to estimating tree biomass, it was natural to apply the ratio approach, thus avoiding similar problems encountered in tree volume estimation (Williams 1982).

The weight-ratio approach uses the following relationship:

$$\hat{w} = \hat{R}\hat{W}$$

where w is merchantable weight, W is total weight, and R is w/W. Interested readers may refer to Honer (1964), Burkhart (1977), and Van Deusen et al. (1981) for the early developmental work on the ratio approach. Parresol and Thomas (1989) compared the weight-ratio approach with the density-integral approach and concluded that the density-integral approach gave more precise estimates of sectional and total bole weight.

Density-integral approach

Parresol and Thomas (1989) derived a tree density-integral model by applying some simple calculus of mass theory. Many standard calculus texts (e.g., Loomis 1977, p. 622; Swokowski 1983, p. 799) define mass (M) for a lamina with a continuous, varying density function $\rho(x, y)$ using a generalized equation:

$$M = \iint_R \rho(x,\ y)\ \mathrm{d}A$$

where R represents the region of integration, A is area, and x and y are the dimensions of the lamina. The three-dimensional structure of a tree is easily conceptualized as a two-dimensional lamina by taking cross-sectional area (y) as one dimension and height above ground (h) as the other dimension (Fig. 1). As we move from lower limit h_l to upper limit h_u in Fig. 1, the variable y is seen to span

$f(h)$. This establishes a basis for adopting the following model for stem biomass:

$$w = \int_{h_l}^{h_u} \int_0^{f(h)} \rho(h, y)\ dy\ dh + e$$

where $f(h)$ is an equation expressing taper in cross-sectional area as a function of height, w is bole dry weight of wood between limits h_l and h_u, and e is residual error. Typically stem profile (cross-sectional area) is modeled using relative height ($x = h/H$) instead of actual height. Performing a change of variable from h to x results in the following generalized stem biomass model:

$$[1] \quad w = H \int_{x_l}^{x_u} \int_0^{f(x)} \rho(x, y)\ dy\ dx + e$$

For a specific biomass model one needs to define ρ and f. In practice, the completely generalized model [1] can be simplified to one integral because specific gravity is usually held constant over the y dimension. In field sampling, disks are cut from bolts for laboratory determination of their specific gravity (SG). Normally a single SG is determined for each disk as opposed to being expressed as a function of radius within the disk. Hence, a tree can be viewed as disks such that the within-disk SG is constant and varies up the tree according to $\rho(x)$. Now stem biomass can be expressed as

$$[2] \quad w = H \int_{x_l}^{x_u} \rho(x)f(x)\ dx + e$$

One could fit stem profile ($f(x)$) and density ($\rho(x)$) independently and place them into model [2] for prediction of biomass. The coefficients of $f(x)$ are of interest in and of themselves because of their application in the prediction of taper and volume. Furthermore, one would expect weight, density, and volume to be correlated at the same measurement bolt on the tree. This correlation is referred to as contemporaneous in econometric methods and literature (Srivastava and Giles 1987, pp. 5–6; Judge et al. 1988, p. 443). It has been shown that ignoring the correlation leads to inefficient estimates of the parameters (see, for example, Srivastava and Giles 1987, Chaps. 3 and 4; Judge et al. 1988, Chap. 11). Thus econometricians recommend to check for contemporaneous correlations.

Data summary

Slash pine

The data are described by Lohrey (1984). Data were collected from permanent growth and yield plots scattered throughout Louisiana. The database consists of 192 felled plantation-grown slash pine (*Pinus elliottii* Engelm. var. *elliottii*) trees, 106 of which were from unthinned stands and 86 of which were from stands that had been thinned from below. All trees were cut at a 0.15-m stump. Diameter inside bark in centimeters was measured at 0.15, 0.6, and 1.4 m and every 1.5 m thereafter throughout the remainder of the stem. Total tree height in meters was measured, and the stem was sectioned into bolts. After each bolt was weighed, a 4 cm thick disk was cut off the bottom end for laboratory determination of wood specific gravity. Bolt volumes inside bark were calculated using Smalian's formula. A weighted (by cross-sectional area) average wood specific gravity was computed based on disks from the upper and lower end of each bolt, so that dry weight of the bolt in kilograms could be determined. Table 1 gives some simple descriptive statistics on the trees.

Table 1. Characteristics of the slash pine plantation trees and willow oak.

	Slash pine		Willow
	Unthinned	Thinned	oak
No. of trees	106	86	61
No. of bolts	1256	1237	1050
Mean age (years)	26	37	68
Age range (years)	12–45	24–48	34–111
Mean DBH (cm)	16.9	26.0	48.7
DBH range (cm)	5.6–33.0	6.9–48.5	17.3–94.0
Mean height (m)	16.2	20.4	27.0
Height range (m)	6–28	9–34	17–38
Mean stem dry wt. (kg)	82	296	1104
Dry wt. range (kg)	2–451	6–1387	88–4056

Willow oak

The willow oak (*Quercus phellos* L.) data are described by Schlaegel (1981); we briefly review his descriptions here. Data were collected from 10 natural bottomland hardwood stands in Mississippi. Sixty-one trees were chosen for destructive sampling from uneven-aged mixed species stands. Stump height, total height, and age were measured on each tree. Diameter inside bark and specific gravity were determined at stump height and then from ground to tree top at 1.5-m intervals along the bole for trees 13 cm DBH and larger and at 0.9-m intervals for trees smaller than 13 cm DBH. Bolt volumes and weights were calculated in an analogous manner to the slash pine. Table 1 gives simple descriptive statistics on the trees.

Models

Taper

Our first step was to obtain a simple taper model that provides a practical fit of the volume of the bole. We settled on a simple trigonometric transformation that resulted in a three-term taper model (Thomas and Parresol 1991) that fit the data as reliably as multitermed polynomial models. It has the form

$$[3] \quad \frac{d^2}{D^2} = \beta_{11}(x - 1) + \beta_{12}(\sin c\pi x) + \beta_{13}\left(\cot \frac{\pi x}{2}\right) + e$$

where d is diameter inside bark, D is DBH, and c is 1.5 for slash pine and 2.0 for willow oak.

Specific gravity

Careful examination of plots of the specific gravity data revealed linear trends over all variables tested. The following variables were used in fitting all possible regressions: relative height, stem diameter inside bark, age, site index (base age 25 years, slash pine only) (Zarnoch and Feduccia

1984), DBH, and total tree height. Examining the residual sum of squares for each subset revealed that only the variables relative height and age were important in estimating bole specific gravity. The following equation was fitted separately to the unthinned and thinned slash plantation data and to the willow oak data:

[4] $\mathrm{SG} = \beta_{20} + \beta_{21}x + \beta_{22}A + e$

where SG is specific gravity, and A represents age in years.

Density integral

The specific integral weight equation for slash pine and willow oak stemwood dry weight is derived below. Given $\rho(x) = (\beta_{20} + \beta_{21}x + \beta_{22}A)k_1$ and $f(x) = D^2[\beta_{11}(x - 1) + \beta_{12}(\sin c\pi x) + \beta_{13}(\cot \pi x/2)]k_2$, where $k_1 = 1000$ (the factor for converting SG to density in kg/m^3) and $k_2 = \pi/40\,000$ (the factor for converting D^2 from cm^2 to area in m^2), the following model is obtained upon integrating eq. 2, simplifying results, and expressing in linear form:

$$[5]\quad \frac{40w}{\pi D^2 H} = \beta_{31}\left[\left(\frac{x_u^2 - x_l^2}{2}\right) - (x_u - x_l)\right] + \beta_{32}A\left[\left(\frac{x_u^2 - x_l^2}{2}\right) - (x_u - x_l)\right] + \beta_{33}\left[\frac{\cos c\pi x_u - \cos c\pi x_l}{c\pi}\right]$$
$$+ \beta_{34}A\left[\frac{\cos c\pi x_u - \cos c\pi x_l}{c\pi}\right] + \beta_{35}\left[\frac{2}{\pi}\ \ln\ \frac{\sin \pi x_u/2}{\sin \pi x_l/2}\right] + \beta_{36}A\left[\frac{2}{\pi}\ \ln\ \frac{\sin \pi x_u/2}{\sin \pi x_l/2}\right]$$
$$+ \beta_{37}\left[\left(\frac{x_u^3 - x_l^3}{3}\right) - \left(\frac{x_u^2 - x_l^2}{2}\right)\right] + \beta_{38}\left[\left(\frac{\sin c\pi x_u - \sin c\pi x_l}{c^2\pi^2}\right) - \left(\frac{x_u \cos c\pi x_u - x_l \cos c\pi x_l}{c\pi}\right)\right]$$
$$+ \beta_{39}\frac{4}{\pi^2}\left[\sin\frac{\pi}{2}x_u - \sin\frac{\pi}{2}x_l + \sum_{k=1}^{\infty}\frac{\prod_{n=1}^{k}(2n - 1)}{\left(\prod_{n=1}^{k}2n\right)(2k + 1)^2}\left[\left(\sin\frac{\pi}{2}x_u\right)^{2k+1} - \left(\sin\frac{\pi}{2}x_l\right)^{2k+1}\right]\right] + e$$

where ln is the natural logarithm, $\beta_{31} = \beta_{11}\beta_{20}$, $\beta_{32} = \beta_{11}\beta_{22}$, $\beta_{33} = -\beta_{12}\beta_{20}$, $\beta_{34} = -\beta_{12}\beta_{22}$, $\beta_{35} = \beta_{13}\beta_{20}$, $\beta_{36} = \beta_{13}\beta_{22}$, $\beta_{37} = \beta_{11}\beta_{21}$, $\beta_{38} = \beta_{12}\beta_{21}$, and $\beta_{39} = \beta_{13}\beta_{21}$. Equations 3, 4, and 5 form a set of linear statistical models with nonlinear cross-equation constraints. If contemporaneous correlations are present and sufficiently large (>0.3, Mehta and Swamy 1976) then seemingly unrelated regressions (SUR) estimation should achieve more efficient parameter estimates than ordinary least squares (OLS).

Estimation

Equations 3, 4, and 5 can be written in the usual matrix algebra notation as

$$\mathbf{y}_1 = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{e}_1$$
$$\mathbf{y}_2 = \mathbf{X}_2\boldsymbol{\beta}_2 + \mathbf{e}_2$$
$$\mathbf{y}_3 = \mathbf{X}_3\boldsymbol{\beta}_3 + \mathbf{e}_3$$

where $\mathbf{y}_i$ is a vector containing the dependent variable from the ith equation, $\mathbf{X}_i$ is a matrix containing the independent variables from the ith equation, and the rest of the notation is self-explanatory. The usual assumptions are

$$E[\mathbf{e}_i] = \mathbf{0} \text{ and } E[\mathbf{e}_i\mathbf{e}_j'] = \sigma_{ij}\mathbf{I}, \qquad i, j = 1, 2, 3$$

Combining all equations yields

$$[6]\quad \underset{(3T \times 1)}{\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \mathbf{y}_3\end{bmatrix}} = \underset{(3T \times 15)}{\begin{bmatrix}\mathbf{X}_1 & 0 & 0\\ 0 & \mathbf{X}_2 & 0\\ 0 & 0 & \mathbf{X}_3\end{bmatrix}}\underset{(15 \times 1)}{\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\\ \boldsymbol{\beta}_3\end{bmatrix}} + \underset{(3T \times 1)}{\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix}}$$

or alternatively

$$\mathbf{y} = \mathbf{f}(\boldsymbol{\beta}) = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$$

where T is number of observations.

The variances and covariances are unknown and must be estimated, with their estimates being used to examine the question of contemporaneous correlations and to form the SUR estimator. To estimate the σ_{ij}, first estimate each equation by least squares $\mathbf{b}_i = (\mathbf{X}_i'\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{y}_i$ and obtain the least squares residuals $\hat{\mathbf{e}}_i = \mathbf{y}_i - \mathbf{X}_i\mathbf{b}_i$. Consistent estimates of the variances and covariances are then given by

$$\hat{\sigma}_{ij} = \frac{1}{(T - K_i)^{1/2}(T - K_j)^{1/2}}\hat{\mathbf{e}}_i'\hat{\mathbf{e}}_j$$

where the degrees of freedom corrections K_i and K_j are the number of coefficients per equation. If we define $\hat{\boldsymbol{\Sigma}}$ as the matrix containing the estimates $\hat{\sigma}_{ij}$, then the restricted SUR estimator is obtained by minimizing $(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \mathbf{I})(\mathbf{y} - \mathbf{X}\beta)$ subject to the nonlinear restrictions $\mathbf{q}(\boldsymbol{\beta}) = \mathbf{0}$. For our purposes, equality restrictions can be introduced by reducing the dimension of the parameter space. The $\boldsymbol{\beta}$ vector contains 15 parameters but as has been noted the system really has only 6 parameters. The restrictions can be expressed in the form

$$\boldsymbol{\beta} = \mathbf{g}(\boldsymbol{\alpha})$$

where $\boldsymbol{\alpha}$ is a six-dimensional vector. Define $\boldsymbol{\alpha} = (\beta_{11},\beta_{12},\beta_{13},\beta_{20},\beta_{21},\beta_{22})'$ and the vector valued function

$$\mathbf{g}(\boldsymbol{\alpha}) = (\beta_{11},\beta_{12},\beta_{13},\beta_{20},\beta_{21},\beta_{22},\beta_{11}\beta_{20},\beta_{11}\beta_{22},-\beta_{12}\beta_{20},$$
$$-\beta_{12}\beta_{22},\beta_{13}\beta_{20},\beta_{13}\beta_{22},\beta_{11}\beta_{21},\beta_{12}\beta_{21},\beta_{13}\beta_{21})'$$

Obtain the restricted SUR estimator by minimizing the objective function $\mathbf{O}(\mathbf{g}(\boldsymbol{\alpha})) = \mathbf{e}'(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \mathbf{I})\mathbf{e}$, where $\mathbf{e} = (\mathbf{y} - \mathbf{X}\mathbf{g}(\boldsymbol{\alpha}))$. Under the Gauss–Newton gradient minimization method, the direction matrix is defined as $\mathbf{F}'\mathbf{F}$, where $\mathbf{F} = [\partial\mathbf{f}/\partial\mathbf{g}(\boldsymbol{\alpha})'|_{\mathbf{g}(\boldsymbol{\alpha})}]$ is the Jacobian matrix (Judge et al. 1985, Appendix B). The iteration function is

$$[7]\quad \hat{\boldsymbol{\alpha}}_{n+1} = \hat{\boldsymbol{\alpha}}_n + [\mathbf{F}_n'(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \mathbf{I})\mathbf{F}_n]^{-1}\mathbf{F}_n'(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \mathbf{I})\hat{\mathbf{e}}_n$$

Table 2. Parameter estimates (with standard errors in parentheses) for the slash pine density-integral system using OLS and SUR.

Parameter	OLS	SUR	% reduction in SE
		Unthinned	
β_{11}	−0.663 (0.003 97)	−0.658 (0.003 30)	17
β_{12}	0.028 2 (0.003 26)	0.021 8 (0.002 47)	24
β_{13}	0.004 41 (0.000 099 3)	0.004 16 (0.000 070 5)	29
β_{20}	0.478 (0.003 82)	0.471 (0.003 34)	13
β_{21}	−0.145 (0.003 90)	−0.145 (0.003 43)	12
β_{22}	0.001 75 (0.000 124)	0.001 87 (0.000 109)	12
		Thinned	
β_{11}	−0.741 (0.003 86)	−0.738 (0.003 27)	15
β_{12}	0.034 4 (0.003 17)	0.026 9 (0.002 54)	20
β_{13}	0.003 78 (0.000 083 9)	0.003 61 (0.000 061 0)	27
β_{20}	0.487 (0.005 09)	0.483 (0.004 47)	12
β_{21}	−0.161 (0.003 39)	−0.157 (0.003 03)	11
β_{22}	0.001 82 (0.000 126)	0.001 77 (0.000 112)	11

Table 3. SUR regression results for the slash pine density-integral system.

	Unthinned stands		Thinned stands	
Model	R^2	RMSE	R^2	RMSE
Taper	0.96	0.056	0.96	0.055
Specific gravity	0.55	0.041	0.66	0.036
Biomass	0.94	0.0022	0.95	0.0018

Note: RMSE, root mean square error.

Starting values for $\hat{\boldsymbol{\alpha}}_1$ were obtained by applying OLS to eqs. 3 and 4. GAUSS version 3.0 matrix language software (Aptech Systems, Inc. 1992) was used to solve for $\hat{\boldsymbol{\alpha}}$ from iteration function [7]. The relative offset orthogonality convergence criterion (ROOCC) of Bates and Watts (1981) was used to determine convergence of the parameter estimates (ROOCC < 0.001). It is defined as

$$\left[\frac{\hat{\mathbf{e}}'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\mathbf{I})\mathbf{F}(\mathbf{F}'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\mathbf{I})\mathbf{F})^{-1}\mathbf{F}'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\mathbf{I})\hat{\mathbf{e}}}{\hat{\mathbf{e}}'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\mathbf{I})\hat{\mathbf{e}})}\right]^{1/2}$$

which is a measure of the orthogonality of the residuals to the Jacobian columns, and goes to zero as the gradient of the objective function becomes small. The covariance matrix of the parameter estimates is calculated as

$$\hat{\boldsymbol{\Sigma}}_{\hat{\alpha}} = [\mathbf{F}'(\hat{\boldsymbol{\Sigma}}^{-1}\otimes\mathbf{I})\mathbf{F}]^{-1}$$

Analysis and results

Testing for contemporaneous correlations

If contemporaneous correlations do not exist, least squares applied to the restricted system in [6] (which is equivalent to applying OLS to models [3] and [4] for this particular system of equations) is fully efficient and there is no need to employ the seemingly unrelated regression estimator. Thus, we want to test whether all of the contemporaneous covariances are zero. It should be noted that only one covariance needs to be nonzero to achieve a theoretical gain in efficiency (Breusch and Pagan 1980). In the context of the density-integral system, the null and alternative hypotheses for this test are

$$H_0\colon \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$

$$H_1\colon \text{at least one covariance is nonzero}$$

An appropriate test statistic is the Lagrange multiplier statistic (Judge et al. 1988, p. 456). In the three-equation case this statistic is given by

$$\lambda = T(r_{12}^2 + r_{13}^2 + r_{23}^2)$$

where r_{ij}^2 is the estimate of squared correlation

$$[8] \quad r_{ij}^2 = \frac{\hat{\sigma}_{ij}^2}{\hat{\sigma}_{ii}\hat{\sigma}_{jj}}$$

Under H_0, λ has an asymptotic χ^2 distribution with 3 df.

Slash pine

Upon applying SUR to the density-integral system of equations for unthinned and thinned slash pine stands, the following $\hat{\boldsymbol{\Sigma}}$ matrices, which contain the $\hat{\sigma}_{ij}$'s, were obtained:

$$\begin{bmatrix} 0.003153 & -0.000165 & 0.0000812 \\ -0.000165 & 0.001711 & 0.0000396 \\ 0.0000812 & 0.0000396 & 4.7083\times10^{-6} \end{bmatrix}_{\text{unthinned}}$$

$$\begin{bmatrix} 0.003050 & 0.0000567 & 0.0000708 \\ 0.0000567 & 0.001277 & 0.0000289 \\ 0.0000708 & 0.0000289 & 3.3673\times10^{-6} \end{bmatrix}_{\text{thinned}}$$

Table 4. Parameter estimates (with standard errors in parentheses) for the willow oak density-integral system using OLS and SUR.

Parameter	OLS	SUR	% reduction in SE
β_{11}	−0.568 (0.004 61)	−0.589 (0.003 76)	18
β_{12}	0.073 6 (0.003 12)	0.074 2 (0.002 50)	20
β_{13}	0.019 6 (0.000 281)	0.017 6 (0.000 236)	16
β_{20}	0.624 (0.003 48)	0.631 (0.003 29)	5
β_{21}	0.063 9 (0.003 24)	0.057 4 (0.003 08)	5
β_{22}	−0.000 730 (0.000 043 7)	−0.000 766 (0.000 041 6)	5

Table 5. SUR regression results for the willow oak density-integral system.

Model	R^2	RMSE
Taper	0.96	0.068
Specific gravity	0.38	0.030
Biomass	0.95	0.0020

Note: RMSE, root mean square error.

Using these values to calculate λ gives

$$\lambda_{\text{unthinned}} = 1256(0.0050 + 0.4445 + 0.1947) = 809.1$$

$$\lambda_{\text{thinned}} = 1237(0.0008 + 0.4878 + 0.1946) = 845.1$$

These large values indicate significant contemporaneous correlations between equations in the system. This suggests SUR should provide gains in efficiency. Table 2 lists the parameter estimates and associated standard errors from OLS and SUR estimation for the slash pine density-integral system. The standard errors are all smaller under SUR estimation, indicating more efficient parameter estimates as was hypothesized. As can be seen in Table 2, standard errors were reduced by 11 to 13% for the specific gravity – biomass coefficients and by 15 to 29% for the taper–biomass coefficients. The contemporaneous correlations between equations can be constructed by taking the square roots of the values from eq. 8 and attaching the sign of the numerator. For the unthinned slash pine system they are $r_{12} = -0.071$, $r_{13} = 0.667$, $r_{23} = 0.441$. For the thinned slash they are $r_{12} = 0.029$, $r_{13} = 0.698$, $r_{23} = 0.441$. There is a strong correlation between the taper (eq. 3) and biomass (eq. 5) models. There is a good correlation between the specific gravity (eq. 4) and the biomass models, and a very weak correlation between taper and specific gravity models. This explains why the standard errors for the specific gravity – biomass coefficients ($\hat{\beta}_{20}$, $\hat{\beta}_{21}$, and $\hat{\beta}_{22}$) were not reduced as much as for the taper–biomass coefficients ($\hat{\beta}_{11}$, $\hat{\beta}_{12}$, and $\hat{\beta}_{13}$). Regression results, in terms of coefficients of determination and root mean square errors, for slash pine are given in Table 3. The fit of the biomass model (eq. 5) appears very good. As a final check of the performance of the SUR approach, we plotted the unthinned and thinned slash pine taper equations and examined the plots to assure that no undue shifts in the shapes of the stem boles had occurred as a result of the fitting procedures.

Willow oak

The results for willow oak are similar. The following $\hat{\Sigma}$ matrix was obtained:

$$\begin{bmatrix} 0.004561 & -0.000113 & 0.000105 \\ -0.000113 & 0.000874 & 0.0000102 \\ 0.000105 & 0.0000102 & 4.136 \times 10^{-6} \end{bmatrix}_{\text{oak}}$$

Using these values to calculate λ gives

$$\lambda_{\text{oak}} = 1050(0.0032 + 0.5880 + 0.0288) = 651.0$$

The Lagrange multiplier statistic indicates significant contemporaneous correlations. Table 4 lists the parameter estimates and associated standard errors from OLS and SUR estimation for the willow oak density-integral system. As with slash pine, the standard errors are all smaller using SUR estimation, indicating that more efficient parameter estimates were achieved over using OLS. As can be seen in Table 4, standard errors were reduced by 5% for the specific gravity – biomass coefficients and by 16 to 20% for the taper–biomass coefficients. The willow oak system did not realize as much of a gain in efficiency as the slash pine systems, which is not surprising because the slash pine grew in homogeneous plantations, whereas the willow oak sample trees were from heterogeneous natural stands. The contemporaneous correlations between the willow oak equations are $r_{12} = -0.057$, $r_{13} = 0.767$, $r_{23} = 0.170$. Again, there is a strong correlation between the taper (eq. 3) and biomass (eq. 5) models. However, the correlation between the specific gravity (eq. 4) and the biomass models is poor, and there is only weak correlation between taper and specific gravity models. Overall, the between-equation correlations are not as strong in willow oak. The poor to weak correlations of the specific gravity error term with the taper and biomass error terms explain why there was only a modest 5% reduction in standard errors for the $\hat{\beta}_{20}$, $\hat{\beta}_{21}$, and $\hat{\beta}_{22}$ coefficients. Table 5 lists the regression results for the willow oak system of equations. As with the slash pine, the fit of the biomass model (eq. 5) appears very good. A plot of the taper equation assured us that the generated stem-profiles matched the data.

A surprising result is that the linear trend of specific gravity over relative tree height ($\hat{\beta}_{21}$) and tree age ($\hat{\beta}_{22}$) had a positive slope and a negative slope, respectively (Table 4). This is counter to what we have observed with every other species we have worked on. Willow oak does not readily prune itself of branches, and a tendency exists for the production of epicormic branches (Burns and Honkala 1990). Perhaps this characteristic promotes an increase in wood density. We have no explanation on why specific gravity should decrease with age, and a graph of the data clearly shows this trend. The mean tree age was 68 years, so these are not young trees. In general, tree growth slows after a certain age with a resultant increase in wood density.

Discussion

Taper and biomass are critical components of forest product characterization. Use of a taper model allows industry to assess the product yields of trees to capture the highest value from the raw wood. Biomass is most important in assessing chip furnish yields for paper industry. The mix of forest products between hardwoods and softwoods has also changed over recent years, making characterization of the yields of hardwoods more widely needed.

The tree taper–volume–biomass system we have developed is based on parsimonious taper models that can be integrated for the whole bole or portions of it and yet have small errors in any portion of the bole. Practically speaking, the functions fit best in that portion of the bole that is merchantable. The deviations that occur are minor, located in the tip and in the stump. Use of the trigonometric function avoids multiple join points in segmented regressions to estimate bole volume. Even so, use of these straightforward trigonometric equations leads to complicated solutions for the entire system. Methods for solution of restricted SUR have provided a route to minimize errors in the coefficients and allow for direct evaluation of the biomass function. The coefficients developed using this approach are important in both high value sawnwood product predictions and in total volume and biomass predictions. Gains in efficiency of estimation compare favorably with other applications of SUR, which also validates the application to this problem (Zellner 1962; Judge et al. 1988, Chap. 11). Taper–volume–biomass data are obtained only through labor-intensive field measurements, which are also costly. The best techniques available should be employed to obtain the best inferences possible. Computers and software have made integrated systems like this possible so that we can take advantage of the mathematically complicated, but unified approach to solutions for the taper–volume–biomass system of equations.

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Appendix A: Integration of x cot $\pi x/2$

Upon evaluating the first integral of model [1] given f is eq. 3 and ρ is eq. 4, along with the appropriate units conversion factors, the following result is obtained

$$w = \frac{\pi}{40} D^2 H \int_{x_l}^{x_u} [\beta_{20} + \beta_{22}A]\left[\beta_{11}(x-1) + \beta_{12}\sin c\pi x + \beta_{13}\cot\frac{\pi}{2}x\right] + \beta_{21}\left[\beta_{11}(x^2 - x) + \beta_{12}x\sin c\pi x + \beta_{13}x\cot\frac{\pi}{2}x\right] dx$$

All the terms in this integral are easily resolved into antiderivatives except for x cot $\pi x/2$. Let $t = \pi x/2$, hence $x = (2/\pi)t$; and let $dt/dx = \pi/2$, hence $dx = (2/\pi)dt$. Substituting we obtain $\int_{t_l}^{t_u}(4t/\pi^2)\cot t\, dt$. Integrating by parts gives

$$\frac{4}{\pi^2}\left[t \ln \sin t - \int_{t_l}^{t_u} \ln \sin t \, dt\right]$$

Let $u = \sin t$, which results in

$$\frac{4}{\pi^2}\left[t \ln u - \int_{\sin t_l}^{\sin t_u} \ln u \frac{du}{\sqrt{1-u^2}}\right]$$

Let $r = \ln u$, $dr = 1/u\, du$, $v = -\cos^{-1} u$, and $dv = du/(1-u^2)^{1/2}$. Again, using integration by parts we obtain

$$\frac{4}{\pi^2}\left[t \ln u - \left(-\cos^{-1} u \ln u - \int_{\sin t_l}^{\sin t_u} - \frac{\cos^{-1} u}{u}\, du\right)\right]$$

By substituting a Maclaurin series expansion for $\cos^{-1} u$ in the integral and integrating each term and combining certain terms we obtain

$$\frac{4}{\pi^2}\left[\ln \sin\frac{\pi}{2}x\left(\frac{\pi}{2}x + \cos^{-1}\sin\frac{\pi}{2}x - \frac{\pi}{2}\right) + \left(u + \frac{u^3}{18} + \frac{3u^5}{200} + \frac{15u^7}{2352} + \ldots\right)\right]_{\sin \pi x_l/2}^{\sin \pi x_u/2}$$

The expression $(\pi x/2 + \cos^{-1} \sin \pi x/2 - \pi/2)$ equals zero, which greatly simplifies the antiderivative expression. Upon evaluating the antiderivative at the limits of integration and writing the result in summation and product notation, the final result is obtained

$$\frac{4}{\pi^2}\left[\sin\frac{\pi}{2}x_u - \sin\frac{\pi}{2}x_l + \sum_{k=1}^{\infty}\frac{\prod_{n=1}^{k}(2n-1)}{\left(\prod_{n=1}^{k}2n\right)(2k+1)^2}\left[\left(\sin\frac{\pi}{2}x_u\right)^{2k+1} - \left(\sin\frac{\pi}{2}x_l\right)^{2k+1}\right]\right]$$

Appendix B: The T_{+m} transformation to accelerate the convergence of infinite series

The T_{+m} transformation (Gray and Clark 1969) is very useful for finding the convergence of an infinite series because it makes the series converge many times faster than normal. This method makes use of the idea of partial sums of a series. S denotes the infinite series and $S(n)$ is the partial sum, of length n, of S. The T_{+m} transformation is defined as

$$\text{[A1]} \quad T_{+m}[S(n+m)] = \frac{S(n)S(n+m-1) - S(n+m)S(n-1)}{S(n) - S(n-1) - S(n+m) + S(n+m-1)}$$

This transformation can be applied to the infinite series expression for the $\int x \cot \pi x/2$. $S(n)$ can be written in terms of S, that is, $S(n) = S + e_n$, where e_n is an error term associated with terms in S that are not in $S(n)$. If the error terms have the property $e_{n+m} \simeq \theta e_n$ then a test for applicabilty and a means of selecting m is through use of the lagged serial correlation coefficient (Jones 1982). The following equation is used for the correlation coefficient

$$\text{[A2]} \quad r = \frac{\sum_{i=1}^{n} S(i)S(i+m) - \frac{\sum_{i=1}^{n} S(i) \sum_{i=1}^{n} S(i+m)}{n}}{\sqrt{\left[\sum_{i=1}^{n} S(i)^2 - \frac{\left(\sum_{i=1}^{n} S(i)\right)^2}{n}\right]\left[\sum_{i=1}^{n} S(i+m)^2 - \frac{\left(\sum_{i=1}^{n} S(i+m)\right)^2}{n}\right]}}$$

This coefficient is tried over different values (lags) of m and the m with the most significant correlation is chosen.

Once a suitable m has been found use eq. A1 and an appropriate stopping rule to find the limit of the sequence. This limit, which is based on the transformation and a partial sum of the original infinite sequence, is as accurate as finding the limit of the infinite sequence but is considerably faster in convergence.

Through a trial process using partial sums $S(1)$ through $S(6)$ and m values of 1, 2, and 3, eq. A2 identified $m = 1$ as the best choice. The following generic BASIC program can be used to find the value of $\int x \cot \pi x/2$:

```
100 REM PROGRAM TO EVALUATE INTEGRAL X * COT(PI/2 * X) USING T+1
110 PRINT TAB(15); "PROGRAM TO EVALUATE INTEGRAL X * COT(PI/2 * X)": PRINT
120 DEFINT I-N: DEFDBL A-H, O-Z
130 DIM S(3)
140 INPUT "LOWER LIMIT OF INTEGRATION ="; XL
150 INPUT "UPPER LIMIT OF INTEGRATION ="; XU
160 PI = 4 * ATN(1)
170 CU = SIN(PI / 2# * XU): CL = SIN(PI / 2# * XL)
180 PROD1 = 1: PROD2 = 1
190 SUM = 0#
200 FOR I = 1 TO 1948 STEP 3
210    FOR K = I TO I + 2
220      PROD1 = PROD1 * 2 * K / 1000:
         PROD2 = PROD2 * (2 * K - 1) / 1000
230      J = 2 * K + 1
240      SUM = SUM + PROD2 * (CU ^ J - CL ^ J) / (PROD1 * J * J)
250      S(K - I + 1) = SUM
260    NEXT K
270    IF I = 1 THEN TOLD = S(3)
280    TPLUS1 = (S(2) * S(2) - S(3) * S(1)) / (2 * S(2) - S(1) - S(3))
290    IF ABS(TPLUS1 - TOLD) < .000000001#
       THEN GOTO 320
300    TOLD = TPLUS1
310 NEXT I
320 VALUE = (2# / PI) ^ 2 * (CU - CL + TPLUS1)
330 PRINT : PRINT TAB(11); "VALUE OF INTEGRAL ="; VALUE, "n+m ="; K - 1
340 END
```