An Alternative View of Continuous Forest Inventories

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Abstract: A generalized three-dimensional concept of continuous forest inventories applicable to all common forest sample designs is presented and discussed. The concept recognizes the forest through time as a three-dimensional population, two dimensions in land area and the third in time. The sample is selected from a finite three-dimensional partitioning of the population. The partitioning is analogous to carving the volume into pieces like a three-dimensional jigsaw puzzle. Each puzzle piece is defined by the selection volumes of observation sets on the individual trees existing in the forest during the period of interest. The concept is a temporal extension of an alternative view of forest sampling offered in Roesch et al. (1993, *Sur. Methods* 19(2):199–204) and results in a finite number of independently selected three-dimensional sample units. FOR. SCI. 54(4):455–464.

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CONTINUOUS FOREST INVENTORIES and monitoring efforts are concerned with evaluating the dynamic state of the targeted forest populations. The inventories rely on a continuous sampling of the forests that can usually be described as a three-part process. First a set of random points is located in two-dimensional space, second a set of times of observation is determined, and third, at each selected time, a cluster of trees in the vicinity of each point is selected for measurement by some rule. Often the set of times of observation define periods of a specific length, initiating in a specific year. In this article, I address the fact that when the determination of the set of observation times is also random and the observations on the sample selected are dependent on the time of observation, the sampled population and the sampling frame are three-dimensional. Such is the case in the US Forest Service's Forest Inventory and Analysis rotating panel sample design (Reams et al. 2005).

Notation

The notational conventions used in this article are as follows. Scalar variables are italicized. If the variable relates to a specific population element it will be upper case and indexed with one or more subscripts defined on the population. If the variable is a sample observation of the population parameter, it will be shown in lower case and indexed with a subscript indicating sample position. Matrices are represented by bold upper case characters. Necessary indices are given in subscripts. Vectors are represented by bold lower case characters. Each vector is defined the first time as either a row or a column vector, depending on its primary use. If a vector is a row or column from a corresponding matrix, the appropriate index is given as a subscript. A transposed vector or matrix is indicated by a prime ($'$). For the reader's convenience, values defined in the article are collected in Table 1.

Area-Based Forest Sampling

A short review of area-based forest sampling will facilitate the discussion. The two most common temporally specific forest sampling rules are (circular, fixed-area) plot sampling and (horizontal) point sampling. In the former, all trees for which the center of the cross-section of the tree bole, at a fixed height above the ground (e.g., 1.37 m), is within a constant horizontal distance (d) of the random point are included in the sample. In the latter, tree i is selected for the sample if the center of the cross-section of the tree bole, at a fixed height above the ground (e.g., 1.37 m), is within a horizontal distance αr_i of the random point, where r_i is the radius of the cross-section (in meters) and α is a constant. The constant α is chosen such that $\alpha^2 = \text{baf}^{-1}$, where baf is the basal area factor, or the number of square meters of basal area per hectare represented by each sample tree. Therefore, tree i is selected with probability proportional to the plot size, πd^2, in plot sampling and with probability proportional to tree basal area, πr_i^2, in point sampling (e.g., see Roesch et al. 1993). By definition, these areal-based designs are two-dimensional because the sample frame partitions a two-dimensional population resulting from a projection of the earth's surface to a plane. There are many variations of these and other areal sampling schemes, some of which can be found in Avery and Burkhart (2002), Shiver and Borders (1996), and Husch et al. (2003).

Roesch et al. (1993) review descriptions of the sample unit in the forestry literature for the various methods of temporally specific forest sampling. For instance, in point sampling the tree is sometimes considered the sample unit (e.g., Oderwald, 1981), whereas at other times the point itself is considered the sample unit (e.g., Husch, 1955). The cluster of trees associated with the point was considered the sample unit in Palley and Horwitz (1961). Roesch et al. (1993) showed an alternative conceptualization that was

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Table 1. Notation used in the manuscript

	Variables
d	Horizontal distance
baf	Basal area factor
α	$\text{baf}^{-1/2}$
r_i	Radius of the cross-section of tree i at a height of 1.37 m
$K_{i,o}$	K-circle of tree i at time o
$Y_{i,o}$	Value of an attribute of interest for tree i at time o
$y_{i,o}$	Observed or measured value of $Y_{i,o}$
$Y_{*,o}$	Total value of interest over all trees at time o
$\tilde{Y}_{*,o}$	Total value of interest at time o across all segments
$\tilde{y}_{s,o}$	Weighted total value of interest at time o for segment s
n_c	Number of cycles
g	Number of panels
P	$n_c g$, length in years of the temporal dimension of the population
A	Area of the areal dimension of the population
V_T	$P * A$, the total areal-temporal volume of the population
$v_{i,o}$	Spatial-temporal sample volume for tree i at time o
$\phi_{i,p}$	Probability that tree i is assigned to panel p
$\pi_{i,p,S}$	Probability of selection for a specific set, **S**, within panel p, of observations on tree i
π_{ip}	Probability of selection for any observation from panel p on tree i
$\pi_{i,p}$	Probability of observing the attributes associated with tree i in year o
$\tilde{p}_{s(i,o)}$	Proportion of $v_{i,o}$ intersecting with population segment s, corresponding to observation set S
Vectors	
$\mathbf{y}_o$	Column vector of length N, containing the time o value of $y_{i,o}$ for each tree i in the population
$\boldsymbol{\phi}_i$	the g-length row vector for tree i of the $\phi_{i,p}$ in order $p = g$ to 1
$\boldsymbol{\pi}_{-,o}$	the column vector extracted from column o *of* $\mathbf{\Pi}$, corresponding to year o
$\boldsymbol{\pi}_{i,-}$	the row vector extracted from row i of $\mathbf{\Pi}$, corresponding to tree i
$\tilde{\boldsymbol{\pi}}$	Column vector of length M, with row s containing the probability of selecting segment s, $\tilde{\pi}_s$
$\mathbf{z}_{i,o}$	Column vector of length M, containing a 1 for each segment that partitions the selection volume for tree i at time o, and a 0 otherwise
$\tilde{\mathbf{p}}_{s(o)}$	Column vector of length N, containing $\tilde{p}_{s(i,o)}$ in position i
$\mathbf{1}_r$	Summation column vector of ones of length equal to the subscript (r)
$\mathbf{w}$	Vector of length M, with each element containing the number of times the corresponding segment s appears in the sample (W_s)
$\ddot{\mathbf{y}}_o$	Vector of length m containing the values of $\tilde{y}_{s,o}$ for each segment in the sample
$\tilde{\mathbf{y}}_o$	Vector of length M containing the values of $\tilde{y}_{s,o}$ for each segment in the population
Matrices	
$\mathbf{V}$	Matrix consisting of N rows and P columns containing $v_{i,o}$ in row i and column o
$\mathbf{\Phi}$	($N \times P$) matrix consisting of n_c copies of ϕ_i in each row i
$\mathbf{\Pi}$	($N \times P$) matrix in which row i, column o contains the probability of observation for tree i at time o
$\ddot{\mathbf{\Pi}}$	($m \times m$) diagonal matrix in which row i, column i contains $\tilde{\pi}_s$ for observation i
$\tilde{\mathbf{\Pi}}$	($N \times N$) diagonal matrix in which row i, column i contains element i from $\tilde{\boldsymbol{\pi}}$
$\tilde{\mathbf{P}}_o$	($N \times M$) matrix with column s containing $\tilde{\mathbf{p}}_{s(o)}$
$\mathbf{w}_o$	($N \times N$) diagonal matrix in which element i, i is the number of times tree i is selected at time o

applicable to all area-based forest sampling schemes. They explain the idea as the jigsaw puzzle view of forest sampling in which the sample units are the mutually exclusive sections of ground resulting from the overlapping selection areas of the individual trees in the forest. These are the same areas as the regions of Palley and O'Regan (1961). Palley and O'Regan (1961), however, treated these regions as elements of the fundamental probability set for selecting clusters of trees, with the population of individual trees comprising the primary population of interest. In the jigsaw puzzle view, the land area is the primary population of interest. The population (or *the puzzle picture*) is partitioned into mutually exclusive, exhaustive sample units (*the puzzle pieces*), which together comprise the sample frame. Figure 1 gives a simple illustration for areal-based sampling, similar to that found in Roesch et al. (1993). In that view, each segment, or puzzle piece, is a sample unit because a random point landing in each segment (a through h in Figure 1) would result in a unique set of observations. A probability sample exists because each puzzle piece has a defined probability of selection, proportional to its size, and the total of these probabilities over all puzzle pieces is equal to 1. This allows the construction of unbiased estimators for any attribute of interest that can be observed in association with the ground segments.

Roesch et al. (1993) also showed that the theory is applicable to remeasured samples for two specific points in time. In Figure 2, the area depicted in Figure 1 is remeasured at a subsequent time from the same points creating a new sampled population or puzzle picture, formed by intersecting the tree selection areas of times 1 and 2. Trees 1, 2, 3, and 4 are centered at their respective numbers. Tree 4 is new at time 2. The solid circles represent the unconditional selection areas of the trees at time 2, and the dashed circles represent the unconditional selection areas at time 1. Each of the segments created by overlapping the two sets of unconditional probabilities is a sample unit. A sample unit created from the joint distribution of time 1 and time 2 probabilities is selected in proportion to its size and is associated with a unique set of measurements jointly spanning times 1 and 2.

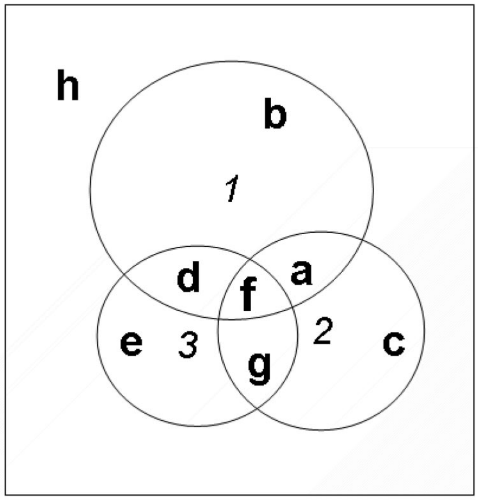

Figure 1. The two-dimensional population as a jigsaw puzzle. Trees 1, 2, and 3 are centered at their respective numbers. The surrounding circles represent the selection areas of the trees. Each lettered segment (puzzle piece) represents a sample unit, selected in proportion to its size.

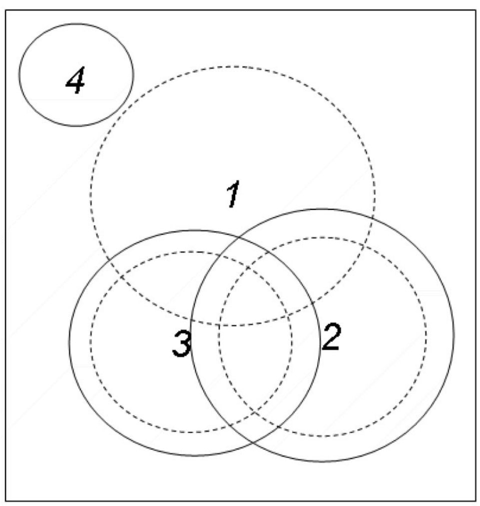

Figure 2. The remeasured sample as a jigsaw puzzle. The same area depicted in Figure 1 is remeasured at a subsequent time from the same points, creating a new sampled population or puzzle picture, formed by intersecting the tree selection areas of times 1 and 2. Trees 1, 2, 3, and 4 are centered at their respective numbers. Tree 4 is new at time 2. The solid circles represent the unconditional selection areas of the trees at time 2, and the dashed circles represent the unconditional selection areas of each tree at time 1. Each of the segments is a sample unit selected in proportion to its size and is associated with a unique set of measurements jointly spanning times 1 and 2.

Area-Based Forest Sampling Through Time

When considering horizontal point samples that have been measured at two specific points in time, Van Deusen et al. (1986, Figure 1) implicitly gave a conditional probability of selection for membership of an individual tree in categories defined by whether it will be measured once or twice. This should give one pause, for it illustrates that if a tree is selected at sample time t_1 from a permanent point and it lives until sample time $t_1 + k$, most forest sampling rules would result in it also being measured at time $t_1 + k$. Therefore, the probability of selection of the tree at time $t_1 + k$ (conditional on its selection at time t_1) is equal to 1, a fact that is of primary interest owing to the dependence on the time t_1 sample. Therefore, successively applying the two-dimensional design while considering only the unconditional, temporally specific, probabilities is erroneous, because new, independent samples are not taken each time. One could attempt to sidestep the issue by arguing that the population is an infinite set of points within two-dimensional space, with a point being the sample unit and therefore the same sample is merely reobserved for associated characteristics. However, when the observations of the associated characteristics are temporally dependent and the times of observation are randomly assigned, the potential sets of observation times must be considered in the definition of the sampled population for a probability sample to exist.

Three-Dimensional Sampling of Forest Populations

Alternatively, one could argue that the set of trees occurring over the area of interest throughout the period of interest is the biological population of interest, rather than the set of trees (and associated measurements) occurring at each specific point in time. To make estimates for the target population, the sampled population must be identifiably associated with the target population. This requires knowledge of the probability of inclusion for the realized set of observations on each tree in the sample over the course of the period of interest. Because there are potentially many sets of observations realizable for each tree in the population, we are led to a reconsideration of the sampled population and the sample unit.

The Three-Dimensional Jigsaw Puzzle View

In this article, I extend the jigsaw puzzle view of Roesch et al. (1993) to include time as a third dimension of the sample frame, facilitating a conceptualization of samples measured at randomly selected times or sets of times. I will show that this conceptualization provides a finite number of mutually exclusive and independently selected sample units for continuous forest inventories. The theory given here is quite robust, whereas a specific heuristic example is given in the Appendix.

The Sample Unit

Each sample unit is a three-dimensional puzzle piece resulting from partitioning the population volume with a solid of revolution for each tree i, created by integrating a_i over time. The individual tree spatial-temporal volumes may be truncated on the sides when adjacent to an areal edge of the population and on the tops and bottoms by time limits of the population. Some designs, such as the panel design discussed below and in the Appendix, will require the union of disjoint pieces of the volume to form a single sample unit. The volume of a sample unit (in area × time units) is divided by the volume of the population to determine the probability of selection for the unit. Although time is a continuous variable, the treatment of time as discrete will often be found to be facilitative. This description ensures a perfect correspondence between the population, sampling frame, and sample unit: the population is divided up into mutually exclusive, exhaustive sample units (*the three-dimensional puzzle pieces*) which in toto comprise the sample frame. Each unit has a definite probability of selection and the total of these probabilities is equal to 1. I will call this the three-dimensional jigsaw puzzle view of continuous forest inventory.

A major advantage of the three-dimensional view as opposed to the two-dimensional view stems from the observation that the inclusion probability arises from a single selection probability that is clearly assigned to each sample unit. Deriving the inclusion probabilities in the two-dimensional view requires recognition of conditional and joint selection probabilities. Any change that would appear to change a selection probability through time in the two-dimensional view actually defines a separate subpopulation or sample unit by the three-dimensional definition. Note that subpopulations may be defined by land area and/or by time.

If time is treated as discrete, the sample unit appears as a set of puzzle pieces created by partitioning the area by overlapping sets of discs, one set for each tree in each panel. In point sampling, the disc size varies through time with tree size, whereas in fixed-area plot sampling the disc size is constant. Time can be partitioned into units of any length. Without loss of generality, I will partition time by a length of 1 year in this article and temporally extend the K-circles of Grosenbaugh and Stover (1957) into cylinders. The K-circle of tree i at time o, $K_{i,o}$, is an imaginary circle, centered at tree center, with radius d in plot sampling and radius $\alpha r_{i,o}$ in point sampling. Palley and O'Regan (1961) clarified earlier work by pointing out that these K-circles were truncated for trees at the edge of the sampled area.

Suppose that there are $\mathbf{N}$ trees, with labels 1, 2, . . . , $\mathbf{N}$, associated with the three-dimensional population of temporal length P starting in year 1. That is, $\mathbf{N}$ is the number of distinct trees that are alive in the land area of interest during at least 1 year of the period of interest. In some inventories, the land area of interest can change over time. This treatise assumes that areas that are not of interest during the entire span of P years constitute separate populations of interest. The spatial-temporal sample volume for tree i at time o, of size $v_{i,o}$ (in acres · year), is the portion of tree i's K-cylinder (or stack of K-cylinders) that is within the population at time o and is the volume from within which a random point will select the associated observations of tree attributes for the sample. To index $v_{i,o}$ annually is equivalent to assuming that $v_{i,o}$ is discrete and remains constant for an entire year. The $v_{i,o}$ can be collected into the matrix,

$$\mathbf{V} = \begin{bmatrix} v_{1,P} & v_{1,P-1} & \cdots & v_{1,1} \\ v_{2,P} & v_{2,P-1} & & \\ \vdots & & \ddots & \vdots \\ v_{N,P} & & \vdots & v_{N,1} \end{bmatrix}.$$

To estimate change over a defined area (A) and temporal period, I assume a continuous forest inventory using a rotating panel design and leave the simplification to a single panel (or nonpaneled design) to the reader. An example of a rotating panel design is in use by the Forest Inventory and Analysis units of the US Forest Service (e.g., see Roesch and Reams, 1999). Designs of this type consist of g mutually exclusive temporal panels. One panel per year is measured for g consecutive years, after which the panel measurement sequence reinitiates. Each complete set of measurements on all panels is often referred to as a cycle. As an example, Table 2 shows the nine potential sets (excluding the null set) of observations on a single live tree over two cycles of a three-panel rotating panel design. Assume that

Table 2. Marginal and conditional probability expressions for the nine possible sets of observations of a single live tree *i* over two cycles (6 years) of an annual rotating panel design consisting of three consecutive panels and a permanent placement of sample points

Year	1	2	3	4	5	6	$\tilde{\boldsymbol{\pi}}$	
$V_T^{-1}\mathbf{V}_{i,-}$	p_1	p_2	p_3	p_4	p_5	p_6		
Panel	1	2	3	1	2	3		
$\boldsymbol{\Phi}_{i,-}$	⅓	⅓	⅓	⅓	⅓	⅓		
Set							Unrestricted case	Increasing probability
1	X						$\tilde{p}_{1(1,1)}\,(p_1/3)$	0
2	X			X			$\tilde{p}_{2(1,1)}\,(p_1/3)$	$(p_1)/3$
3				X			$((1-p_1)(p(Y_4\|\tilde{Y}_1)))/3$	$(p_4-p_1)/3$
4		X					$p_2(1-p(Y_5\|Y_2))/3$	0
5		X			X		$p_2(p(Y_5\|Y_2))/3$	$(p_2)/3$
6					X		$((1-p_2)(p(Y_5\|\tilde{Y}_2)))/3$	$(p_5-p_2)/3$
7			X				$p_3(1-p(Y_6\|Y_3))/3$	0
8			X			X	$p_3(p(Y_6\|Y_3))/3$	$(p_6)/3$
9						X	$((1-p_3)(p(Y_6\|\tilde{Y}_3)))/3$	$(p_6-p_3)/3$
$\boldsymbol{\pi}_{i,-}$	$p_1/3$	$p_2/3$	$p_3/3$	$p_4/3$	$p_5/3$	$p_6/3$		
Probability of observation during cycle		$(p_1+p_2+p_3)/3$ Cycle 1			$(p_4+p_5+p_6)/3$ Cycle 2			

Matrices from the text are subscripted to indicate a single row for tree *i*. The case of increasing probability assumes that a tree has positive growth from year to year and is selected with probability proportional to size. The case of unrestricted probability assumes that negative growth is possible, and the tree is observed with probability proportional to size. Set refers to the collection of years of observation. The "Cycle 1" observation could be made during years 1, 2, or 3. The "Cycle 2" observation could be made during years 4, 5, or 6.

the continuous inventory consists of n_c cycles and therefore $P = n_c g$ years. That is, if panel 1 is measured in year y, it will also be measured in years $y + g$, $y + 2g$, and so on, through to year $y + (n_c - 1)g$. Panel 2 would then be measured in years $y + 1$, $y + 1 + g$, $y + 1 + 2g$, etc.

To define the matrix of selection probabilities, we must incorporate the probability of each tree's assignment to a potential measurement panel. Represent the probability that tree i is assigned to potential measurement in panel p as $\phi_{i,p}$ and ensure that $\sum_{p=1}^{g} \phi_{i,p} = 1$. Although there are a number of ways that this can be done, the simplest way would be to divide the land area into "panels" using a randomly-placed grid with cells of known size, and assign trees to panels based on their basal centers. For each tree, form a g-length row vector $\boldsymbol{\phi}_i$ of the $\phi_{i,p}$ in order $p = g$ to 1. Concatenate n_c copies of $\boldsymbol{\phi}_i$ into an $n_c g$-length row vector. Order the **N** resulting row vectors from 1 to N to form the matrix $\boldsymbol{\Phi}$. Often, the rows of $\boldsymbol{\Phi}$ will be identical, as would be the case if panel assignment were area-based. If it were further true that a tree had the same probability of being assigned to each panel, then the elements of $\boldsymbol{\Phi}$ would all be equal. In all cases, the elements of each row will sum to n_c, and the elements over all rows and all columns will sum to $n_c N$. The unconditional probabilities of observation owing to a random point in three-dimensional spatial-temporal volume can be then be represented by the matrix,

$$\boldsymbol{\Pi} = V_{\mathrm{T}}^{-1}(\boldsymbol{\Phi} \cdot\!* \, \mathbf{V})$$
$$= \begin{bmatrix} \pi_{1,n_c g} & \pi_{1,(g-1)+(n_c-1)g} & \cdots & \pi_{1,1} \\ \pi_{2,n_c g} & \pi_{2,(g-1)+(n_c-1)g} & & \\ \vdots & & \ddots & \vdots \\ \pi_{N,n_c g} & & \vdots & \pi_{N,1} \end{bmatrix}$$

where V_{T} is the total areal-temporal volume of the population, and $\cdot *$ indicates element-by-element matrix multiplication, that is, each element (i, p) in $\boldsymbol{\Phi}$ is multiplied by the corresponding element (i, p) in $\mathbf{V}$. A vector equal to a specific column of $\boldsymbol{\Pi}$, corresponding to a specific year o will be denoted $\pi_{-,\,o}$. A vector equal to a specific row of $\boldsymbol{\Pi}$, corresponding to a specific tree i will be denoted $\pi_{i,\,-}$.

Often, the probability of selection for a specific temporal set of observations on the tree is of interest; that set is a subset of all potential observation times within the panel. By the most common forest sampling rules, there are potentially n_c sets of observations on each tree for each panel. Using the standard notation of probability theory as can be found in Olkin et al. (1980), the probability of selection, by a random point in three-dimensional space for a specific set **S** within panel p, of observations on tree i ($\pi_{i,o,\mathbf{S}}$) is $\pi_{i,p,\mathbf{S}} = \phi_{i,p} p(\mathbf{S}) = \bigcap_{o=1}^{n_\mathrm{S}} \pi_{i,o\in\mathbf{S}}$, where $p(\mathbf{S})$ is the probability that set **S** of panel p is selected and n_s is the number of observations in the set.

The joint probability of selection, by a random point in three-dimensional space, for a set, $\boldsymbol{S}_1$, of observations (indexed by time of observation o) on tree i and a set $\boldsymbol{S}_2$ on tree j is

$$\pi_{i\mathbf{S}_1,j\mathbf{S}_2} = \bigcap_{i,j} \left(\bigcap_{o=1}^{n_{\mathbf{S}_1}} \pi_{\mathrm{i,o}\in\mathbf{S}_1},\ \bigcap_{o=1}^{n_{\mathbf{S}_2}} \pi_{\mathrm{j,o}\in\mathbf{S}_2} \right).$$

Assuming that tree centers do not move through time $\pi_{i,p,\mathbf{S}} = \min(\pi_{i,o\in\mathbf{S}})$. If we further assume that the selection areas do not shrink: $\pi_{i,p,\mathbf{S}} = \pi_{i,\min(o)\in\mathbf{S}}$. For completeness, express the probability of having made any observation from panel p on tree i as $\pi_{ip} = \bigcup_{\mathrm{S}=1}^{n_{ip\mathrm{S}}} \pi_{i,p,\mathrm{S}}$, where $n_{ip}\mathbf{s}$ is the number of sets of observations on tree i within panel p.

When one considers all of the possible intersections of all observation sets for all trees associated with the population, one has fully defined all sample units and the sample frame. That is, the spatial-temporal volume (or statue) has been carved into pieces that are selected with probability proportional to their size. Each piece is associated with a unique set of observation times on trees in the forest.

A random point selected from the surface of a forest can be used with any function to observe tree attributes through time. If the function is temporally dependent, then one must integrate over time to determine a probability of inclusion. The probability of observing the attributes associated with tree i in year o is $\pi_{i,o} = \tilde{\boldsymbol{\pi}}'\mathbf{z}_{i,o}$, where $\mathbf{z}_{i,o}$ is an indicator column vector of length M (the number of segments in the population), containing a 1 in the position corresponding to each segment s that partitions the selection volume for tree i at time o and a 0 otherwise; and $\tilde{\boldsymbol{\pi}}$ is a column vector of length M, with each row containing the probability of selecting each segment π_s, corresponding to observation set $\boldsymbol{S}$. For later use, I also define the $m \times m$ diagonal matrix $\tilde{\boldsymbol{\Pi}}$ containing element s from $\tilde{\boldsymbol{\pi}}$ in position (s, s) and zeros elsewhere.

Let $\tilde{p}_{s(i,o)}$ be the proportion of $v_{i,o}$ intersecting with population segment s, corresponding to observation set $\boldsymbol{S}$. Note that

$$\tilde{p}_{s(i,o)} = \begin{cases} \dfrac{\tilde{\pi}_s}{\pi_{i,o}} & \text{if tree } i \text{ can be observed from segment } s \text{ at time } o \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Also, note that the sum over all segments s of $\tilde{p}_{s(i,o)}$ is 1. For each segment at each time of observation, collect the N values of $\tilde{p}_{s(i,o)}$ into a column vector of length N, by the index i, denoted $\tilde{\mathbf{p}}_{s(o)}$. Let $Y_{i,o}$ represent the value of an attribute of interest for tree i at time o, and let $y_{i,o}$ represent the sample measurement or observation of $Y_{i,o}$. For time o, collect the N values of $Y_{i,o}$ into a column vector of length N, by the index i, denoted $\mathbf{y}_o$. I can now write a temporally specific observation for each segment as a sum of weighted tree values:

$$\tilde{y}_{s,o} = \tilde{\mathbf{p}}_{s(o)}{}'\mathbf{y}_o. \quad (1)$$

Represent a column vector of ones of length equal to its subscript r as $\mathbf{1}_r$. The temporally specific total value of interest over all trees is then ($Y_{*,o} = \mathbf{1}_N'\mathbf{y}_\mathrm{o}$). For simplicity, assume that the focus of interest is on the estimation of $Y_{*,o}$, using m random points to select m segments from the population volume with the same assumptions as above (that is, sampling is with replacement). Collect the m sample values of $\tilde{\pi}_s^{-1}$ into an $m \times m$ diagonal matrix, denoted $\ddot{\mathbf{\Pi}}^{-1}$. Collect the m sample observations of $\tilde{y}_{s,o}$ into a column vector of length m, denoted $\ddot{\mathbf{y}}_o$.

A naive estimator for attributes at a specific time o is one that is calculated without the benefit of strength (or knowledge) drawn from observations on other panels and is often used to initialize more efficient estimators (e.g., Roesch, 2007). An unbiased, naive estimator of $Y_{*,o}$ for this sample, selected with probability proportional to size, is

$$\hat{Y}_{*,o} = m^{-1}\ddot{\mathbf{\Pi}}^{-1}\ddot{\mathbf{y}}_o = m^{-1}(\mathbf{w}'\tilde{\mathbf{\Pi}}^{-1})\tilde{\mathbf{y}}_o. \quad (2)$$

where $\mathbf{w}$ is a column vector of length M, with each element containing W_S, the number of times the corresponding segment s appears in the sample, and $\tilde{\mathbf{y}}_o$ is a vector of length M containing the time o values of $\tilde{y}_{s,o}$ for each segment in the population. Note that W_S is a random integer between 0 and m, inclusive, and all other quantities are fixed.

I represent a column vector of ones of length M as $\mathbf{1}_M$ and define the total time o value of interest across all segments as $\tilde{Y}_{*,o} = \tilde{\mathbf{y}}_o'\mathbf{1}_M$. As in Roesch et al. (1993), to show that $\hat{Y}_{*,o}$ is an unbiased estimator of $Y_{*,o}$, I will first show $\hat{Y}_{*,o}$ to be unbiased for $\tilde{Y}_{*,o}$ and then show that $\tilde{Y}_{*,o}$ equals $Y_{*,o}$. Following Cochran (1977, p. 252–255), I can show $\hat{Y}_{*,o}$ to be unbiased for $\tilde{Y}_{*,o}$:

$$\begin{aligned} \mathrm{E}[\hat{Y}_{*,o}] &= \mathrm{E}[m^{-1}(\mathbf{w}'\tilde{\mathbf{\Pi}}^{-1})\tilde{\mathbf{y}}_o] \\ &= m^{-1}\tilde{\mathbf{y}}_o'\,\mathrm{E}[\tilde{\mathbf{\Pi}}^{-1}\mathbf{w}] \\ &= m^{-1}\tilde{\mathbf{y}}_o'\{\tilde{\mathbf{\Pi}}^{-1}(\mathbf{E}[\mathbf{w}])\}. \end{aligned}$$

Because $\mathbf{w}$ is a vector of multinomial random variables, with expected value equal to $m\tilde{\boldsymbol{\pi}}$, the result is

$$\begin{aligned} \mathrm{E}[\hat{Y}_{*,o}] &= m^{-1}\tilde{\mathbf{y}}_o'\{\tilde{\mathbf{\Pi}}^{-1}(m\tilde{\pi})\} \\ &= (m^{-1}m)\tilde{\mathbf{y}}_o'\{\tilde{\mathbf{\Pi}}^{-1}\tilde{\boldsymbol{\pi}}\} \\ &= \tilde{\mathbf{y}}_o'\mathbf{1}_M \\ &= \tilde{Y}_{*,o}. \end{aligned}$$

Substituting the right-hand side of Equation 1 for $\tilde{y}_{s,o}$ in the definition of $\tilde{Y}_{s,o}$ and collecting the M column vectors ($\tilde{\mathbf{p}}_{s(o)}$) into the N row by M column matrix $\tilde{\mathbf{P}}_o$:

$$\tilde{\mathbf{Y}}_{*,o} = \tilde{\mathbf{y}}_o'\mathbf{1}_M = (\tilde{\mathbf{P}}_o'\mathbf{y}_o)'\mathbf{1}_M. \quad (3)$$

Therefore,

$$\begin{aligned} \tilde{Y}_{*,o} &= \mathbf{y}_o'\tilde{\mathbf{P}}_o\mathbf{1}_M \\ &= \mathbf{y}_o'\mathbf{1}_N \\ &= Y_{*,o} \quad \text{Q.E.D.} \end{aligned} \quad (4)$$

Expanding Equation 2 to include the definition of $\tilde{\mathbf{y}}_o$ and subsequent rearrangement gives

$$\begin{aligned} \hat{Y}_{*,o} &= m^{-1}(\mathbf{w}'\tilde{\mathbf{\Pi}}^{-1})\tilde{\mathbf{y}}_o \\ &= m^{-1}(\mathbf{w}'\tilde{\mathbf{\Pi}}^{-1})\,(\tilde{\mathbf{P}}_o'\mathbf{y}_o) \\ &= m^{-1}(\mathbf{w}'\tilde{\mathbf{\Pi}}^{-1}\tilde{\mathbf{P}}_o')\mathbf{y}_o \\ &= m^{-1}(\mathbf{W}_o'\boldsymbol{\pi}_{-,o}^{-1})'\mathbf{y}_o \end{aligned} \quad (5)$$

where $\mathbf{W}_o$ is an $N \times N$ diagonal matrix in which diagonal element i, i equals the number of times tree i is selected for observation at time o and $\boldsymbol{\pi}_{-,o}^{-1}$ is an N × 1 column vector with each element being the inverse of the corresponding element of $\boldsymbol{\pi}_{-,o}$, previously defined as column o of $\mathbf{\Pi}$. The final expression in Equation 5 is the three-dimensional probability proportional to size estimator, in that it uses the three-dimensional probability of inclusion to formulate temporally specific estimates. The variance of $\hat{Y}_{*,o}$ is

$$\begin{aligned} \mathrm{Var}(\hat{Y}_{*,o}) &= m^{-1}(\tilde{\mathbf{\Pi}}(\tilde{\mathbf{\Pi}}^{-1}\tilde{\mathbf{y}}_o - \tilde{Y}_{*,o} * \mathbf{1}_M))' \\ &\quad \times (\tilde{\mathbf{\Pi}}^{-1}\tilde{\mathbf{y}}_o - \tilde{Y}_{*,o} * \mathbf{1}_M) \\ &= m^{-1}(\tilde{\mathbf{y}}_o - \tilde{Y}_{*,o} * \tilde{\boldsymbol{\pi}})'(\tilde{\mathbf{\Pi}}^{-1}\tilde{\mathbf{y}}_o - \tilde{Y}_{*,o} * \mathbf{1}_M). \end{aligned} \quad (6)$$

The sample estimate of the variance is then (Cochran, 1977)

$$\begin{aligned} \mathrm{var}(\hat{Y}_{*,o}) &= m^{-1}(m-1)^{-1}(\ddot{\mathbf{\Pi}}^{-1}\ddot{\mathbf{y}}_o - \hat{Y}_{*,o} * \mathbf{1}_m)' \\ &\quad \times (\ddot{\mathbf{\Pi}}^{-1}\ddot{\mathbf{y}}_o - \hat{Y}_{*,o} * \mathbf{1}_m). \end{aligned} \quad (7)$$

Conclusion

The description of continuous forest inventories as a sample of a three-dimensional finite population given above is uniquely informative. It arose from the recognition of the importance of the time of observation on the outcome of the sample. This finite population view is useful in at least the same two ways that the two-dimensional jigsaw puzzle view was considered useful for forest inventories by Roesch et al. (1993). Those uses were education and the construction of sampling simulations to test sample design parameters and estimation methods. With respect to education, students should readily grasp the idea that regardless of the method used to determine the observation points in space or time (e.g., remeasured plot sampling or point sampling, with or without temporal panels), all continuous forest inventory schemes could be thought of as cutting the puzzle volume up into pieces and selecting the pieces with probability proportional to their size. The simulation advantage is obvious: one must simply define all of the sample units as well as their associated attributes of interest and then select from the finite number of units as many times as one chooses, rather than selecting from an infinite population of points and determining all of the attributes associated with each point drawn in each and every iteration.

Another very practical advantage of this three-dimensional jigsaw puzzle view of continuous forest inventories is that it clearly defines the database structure necessary for any specific spatial-temporal design. That is, the general development above in Equations 1 through 7 can applied to any areal-temporal forest sampling design to determine the appropriate database structure for subsequent analyses. Although a thorough discussion of database design is beyond the scope of this article, the point is simple: databases work best when they are organized in accordance with the most commonly performed tasks. When temporal partitioning is required to calculate inclusion probabilities for estimation from a continuous forest inventory, then a database that uses temporal partitions as a dimension, rather than as a field or group of fields in the areal dimension will usually be more efficient.

Literature Cited

AVERY, T. E., AND H. E. BURKHART. 2002. *Forest measurements*, 5th ed. McGraw-Hill, New York, NY. 456 p.

COCHRAN, W. G. 1977. *Sampling techniques*, 3rd ed. John Wiley & Sons, New York, NY. 428 p.

GROSENBAUGH, L. R., AND W. S. STOVER. 1957. Point-sampling compared with plot-sampling in southeast Texas. *For. Sci.* 3(1):2–14.

HUSCH, B. 1955. Results of an investigation of the variable plot method of cruising. *J. For.* 53:570–574.

HUSCH, B., T. W. BEERS, AND J. A. KERSHAW, JR. 2003. *Forest mensuration*, 4th ed. John Wiley & Sons, New York, NY. 443 p.

OLKIN, I., L. J. GLESER, AND C. DERMAN. 1980. *Probability models and applications.* Macmillan Publishing Company, New York, NY. xiii and 576 p.

ODERWALD, R. G. 1981. Point and plot sampling—The relationship. *J. For.* 79:377–378.

PALLEY, M. N., AND L. G. HORWITZ. 1961. Properties of some random and systematic point sampling estimators. *For. Sci.* 7(1):52–65.

PALLEY, M. N., AND W. G. O'REGAN. 1961. A computer technique for the study of forest sampling methods. I. Point sampling compared with line sampling. *For. Sci.* 7(3):282:294.

REAMS, G. A., W. D. SMITH, M. H. HANSEN, W. A. BECHTOLD, F. A. ROESCH, AND G. G. MOISEN. 2005. The forest inventory and analysis sampling frame. P. 11–26 in *The enhanced forest inventory and analysis program-national sampling design and estimation procedures*, Bechtold, W. A. and P. L. Patterson (eds.). Gen. Tech. Rep. SRS-80. U.S. Forest Service, Asheville, NC. 85 pp.

ROESCH, F. A. 2007. Compatible estimators of the components of change for a rotating panel forest inventory design. *For. Sci.* 53(1):50–61.

ROESCH, F. A., JR., E. J. GREEN, AND C. T. SCOTT. 1993. An alternative view of forest sampling. *Surv. Methodol.* 19(2):199–204.

ROESCH, F. A., AND G. A. REAMS. 1999. Analytical alternatives for an annual inventory system. *J. For.* 97(12):33–37.

SHIVER, B. D., AND B. E. BORDERS. 1996. *Sampling techniques for forest resource inventory.* John Wiley & Sons, New York, NY. 356 p.

VAN DEUSEN, P. C., T. R. DELL, AND C. E. THOMAS. 1986. Volume growth estimation from permanent horizontal points. *For. Sci.* 32(2):415–422.

Appendix

The following is a heuristic example of a population and sample matrices that arise from it. Assume that we have a population consisting of $N = 5$ trees labeled A, B, C, D, and E, respectively, in a square area of 40 m on a side, over a period of $P = 10$ years. Each tree occupies a row in an attribute matrix, and each year occupies a column. Tree A is alive and in the population for the entire period. Tree B enters during year 3 and survives for the rest of the period. Tree C enters during year 1, survives through year 9, and dies during year 10. Tree D is alive at the beginning of the period and harvested during year 4. Tree E enters during year 10. Define an attribute matrix, consisting of the collected vectors, $\mathbf{y}_{i,o}$, of attributes for each tree i at each time o:

$$\begin{array}{cc} \text{year} = & \begin{array}{cccccc} 10 & 9 & \cdots & 3 & 2 & 1 \end{array} \quad \text{Tree} \\ \mathbf{Y} = & \begin{bmatrix} \mathbf{y}_{A,10} & \mathbf{y}_{A,9} & \cdots & \mathbf{y}_{A,3} & \mathbf{y}_{A,2} & \mathbf{y}_{A,1} \\ \mathbf{y}_{B,10} & \mathbf{y}_{B,9} & \cdots & \mathbf{y}_{B,3} & 0 & 0 \\ 0 & \mathbf{y}_{C,9} & \cdots & \mathbf{y}_{C,3} & \mathbf{y}_{C,2} & \mathbf{y}_{C,1} \\ 0 & 0 & \cdots & \mathbf{y}_{D,3} & \mathbf{y}_{D,2} & \mathbf{y}_{D,1} \\ \mathbf{y}_{E,10} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}. \end{array}$$

Assume that horizontal point sampling with a basal area factor of 3 m^2/ha will be used to determine sample trees for measurement in a continuous forest inventory using an annually rotating panel design in which there are five panels, each applied with equal probability. If tree radius is in cm, then $\alpha = 0.57735$. Therefore,

$$\mathbf{V} = \pi\alpha^2 * \begin{bmatrix} r^2_{A,10} & r^2_{A,9} & \cdots & r^2_{A,3} & r^2_{A,2} & r^2_{A,1} \\ r^2_{B,10} & r^2_{B,9} & \cdots & r^2_{B,3} & 0 & 0 \\ 0 & r^2_{C,9} & \cdots & r^2_{C,3} & r^2_{C,2} & r^2_{C,1} \\ 0 & 0 & \cdots & r^2_{D,3} & r^2_{D,2} & r^2_{D,1} \\ r^2_{E,10} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} v_{A,10} & v_{A,9} & \cdots & v_{A,3} & v_{A,2} & v_{A,1} \\ v_{B,10} & v_{B,9} & \cdots & v_{B,3} & 0 & 0 \\ 0 & v_{C,9} & \cdots & v_{C,3} & v_{C,2} & v_{C,1} \\ 0 & 0 & \cdots & v_{D,3} & v_{D,2} & v_{D,1} \\ v_{E,10} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

For $p = 1$ to $5\phi_{i,p} = 0.2$, and $\boldsymbol{\varphi}_i = [0.2\ \ 0.2\ \ 0.2\ \ 0.2\ \ 0.2]$, and

$$\boldsymbol{\Phi} = \begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix}.$$

The probability matrix of observing the attributes of each tree i during each year o is

$$\mathbf{\Pi} = V_T^{-1}(\mathbf{\Phi} \cdot * \mathbf{V})$$

$$= \begin{bmatrix} 0.2V_T^{-1}\alpha r_{A,10} & 0.2V_T^{-1}\alpha r_{A,9} & \cdots & 0.2V_T^{-1}\alpha r_{A,3} & 0.2V_T^{-1}\alpha r_{A,2} & 0.2V_V^{-1}\alpha r_{A,1} \\ 0.2V_T^{-1}\alpha r_{B,10} & 0.2V_T^{-1}\alpha r_{B,9} & \cdots & 0.2V_T^{-1}\alpha r_{B,3} & 0 & 0 \\ 0 & 0.2V_T^{-1}\alpha r_{C,9} & \cdots & 0.2V_T^{-1}\alpha r_{C,3} & 0.2V_T^{-1}\alpha r_{C,2} & 0.2V_T^{-1}\alpha r_{C,1} \\ 0 & 0 & \cdots & 0.2V_T^{-1}\alpha r_{D,3} & 0.2V_T^{-1}\alpha r_{D,2} & 0.2V_T^{-1}\alpha r_{D,3} \\ 0.2V_T^{-1}\alpha r_{E,10} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \pi_{A,10} & \pi_{A,9} & \cdots & \pi_{A,3} & \pi_{A,2} & \pi_{A,1} \\ \pi_{B,10} & \pi_{B,9} & \cdots & \pi_{B,3} & 0 & 0 \\ 0 & \pi_{C,9} & \cdots & \pi_{C,3} & \pi_{C,2} & \pi_{C,1} \\ 0 & 0 & \cdots & \pi_{D,3} & \pi_{D,2} & \pi_{D,1} \\ \pi_{E,10} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

This particular design has 15 potential temporal partitions of the population for the observation sets. These include the five partitions (T_1, T_2, T_3, T_4, and T_5, below) corresponding to the respective events ($\{E_{1,6}\}$, $\{E_{2,7}\}$, $\{E_{3,8}\}$, $\{E_{4,9}\}$, and $\{E_{5,10}\}$, where the subscripts indicate years of panel measurement) that a tree (or set of trees) is observed in both years of each panel, and the 10 partitions (T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, and T_{15}) corresponding to the respective events ($\{E_1\}$, $\{E_2\}$, $\{E_3\}$, $\{E_4\}$, $\{E_5\}$, $\{E_6\}$, $\{E_7\}$, $\{E_8\}$, $\{E_9\}$, and $\{E_{10}\}$) that a tree (or set of trees) is observed in only one of the subscripted population years 1 through 10. The 15 temporal panels into which the population is divided are then

$$\begin{array}{lll} T_1 = \{E_{1,6}\} & T_6 = \{E_1\} & T_{11} = \{E_6\} \\ T_2 = \{E_{2,7}\} & T_7 = \{E_2\} & T_{12} = \{E_7\} \\ T_3 = \{E_{3,8}\} & T_8 = \{E_3\} & T_{13} = \{E_8\} \\ T_4 = \{E_{4,9}\} & T_9 = \{E_4\} & T_{14} = \{E_9\} \\ T_5 = \{E_{5,10}\} & T_{10} = \{E_5\} & T_{15} = \{E_{10}\} \end{array}$$

Additionally, a tree (or set of trees) might not be observed at all. If all trees had been alive for all periods, this would lead to $\binom{16}{1}^5 =$ 1,048,576 potential observation sets. However, in this example tree A is eligible for all 15 temporal partitions, tree B for 11 temporal partitions, tree C for 13 temporal partitions, tree D for 3 temporal partitions, and tree E for 1 temporal partition, leading to

$$\binom{16}{1}\binom{12}{1}\binom{14}{1}\binom{4}{1}\binom{2}{1} = 21{,}504 \text{ potential combinations.}$$

Under the common sampling rules, we can assume that once a tree is measured, it will be measured at each successive occasion as long as it exists. Under this assumption tree A is eligible for 10 temporal partitions, tree B for 8 temporal partitions, tree C for 9 temporal partitions, tree D for 3 temporal partitions, and tree E for 1 temporal partition, which leads to at most

$$\binom{11}{1}\binom{9}{1}\binom{10}{1}\binom{4}{1}\binom{2}{1} = 7{,}920 \text{ potential combinations.}$$

Knowledge of the spatial clustering of the trees is necessary to define the potential sets of observations.

Under the same assumptions as above, recall that tree A is eligible for 10 temporal events, tree B for 8 temporal events, tree C for 9 temporal events, tree D for 3 temporal events, and tree E for 1 temporal event, The sets of tree observations potentially associated with each of the 15 temporal events would then be

$$S(T_1) \subset \{A_{1,6}, C_{1,6}, AC_{1,6}, \varnothing_{1,6}\}$$

$$S(T_2) \subset \{A_{2,7}, C_{2,7}, AC_{2,7}, \varnothing_{2,7}\}$$

$$S(T_3) \subset \{A_{3,8}, B_{3,8}, C_{3,8}, AB_{3,8}, AC_{3,8}, BC_{3,8}, ABC_{3,8}, \varnothing_{3,8}\}$$

$$S(T_4) \subset \{A_{4,9}, B_{4,9}, C_{4,9}, AB_{4,9}, AC_{4,9}, BC_{4,9}, ABC_{4,9}, \varnothing_{4,9}\}$$

$$S(T_5) \subset \{A_{5,10}, B_{5,10}, AB_{5,10}, \varnothing_{5,10}\}$$

$$S(T_6) \subset \{D_1, \varnothing_1\}$$

$$S(T_7) \subset \{D_2, \varnothing_2\}$$

$$S(T_8) \subset \{D_3, \varnothing_3\}$$

$$S(T_9) \subset \{\varnothing_4\}$$

$$S(T_{10}) \subset \{C_5, \varnothing_5\}$$

$$S(T_{11}) \subset \{A_6, B_6, C_6, AB_6, AC_6, BC_6, ABC_6, \varnothing_6\}$$

$$S(T_{12}) \subset \{A_7, B_7, C_7, AB_7, AC_7, BC_7, ABC_7, \varnothing_7\}$$

$$S(T_{13}) \subset \{A_8, B_8, C_8, AB_8, AC_8, BC_8, ABC_8, \varnothing_8\}$$

$$S(T_{14}) \subset \{A_9, B_9, C_9, AB_9, AC_9\ BC_9, ABC_9, \varnothing_9\}$$

$$S(T_{15}) \subset \{A_{10}, B_{10}, E_{10}, AB_{10}, AE_{10}, BE_{10}, ABE_{10}, \varnothing_{10}\}$$

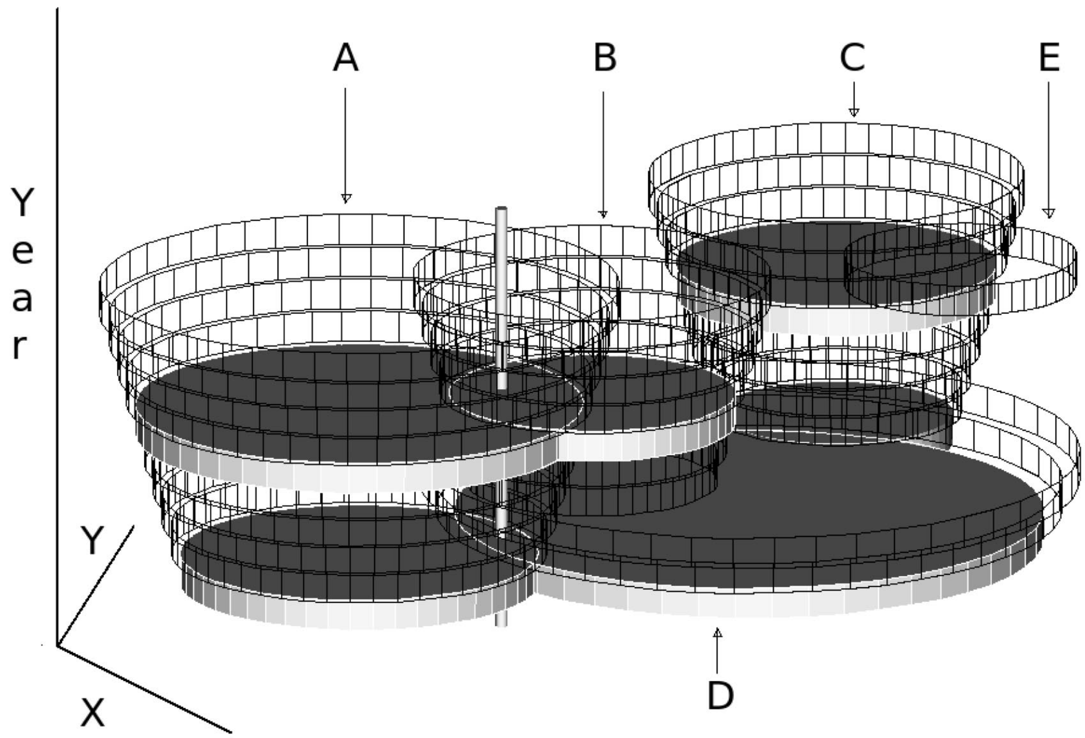

Figure A1. The probability volumes within the three-dimensional population described in this Appendix. The population consists of potential observation sets on five trees labeled A, B, C, D and E, in a 1,600 m^2 area over a period of 10 years. The figure highlights a panel 1 selection, which is specific to years 1 and 6. Depending on the context, the joint, marginal, or conditional probabilities may be of interest. The single ray passing through time and partitioned by panel is "selecting" $A_{1,6}$, D_1, and B_6. Note that $A_{1,6}$ is selected from temporal partition $S(T_1)$, whereas D_1 is selected from temporal partition $S(T_6)$ and B_6 is selected from $S(T_{11})$, all described in the text.

where the letters within brackets indicate the tree or combination of trees potentially observed at the subscripted times relevant to the temporal partition (one or both times of potential panel measurement). Conventionally, the empty set is indicated by $\varnothing$, which is also subscripted to indicate the times relevant to the temporal partition.

Given the potential observation sets above, for the five-tree population, this results in at most

$$3\left[\binom{2}{2}+\binom{2}{1}+\binom{2}{0}\right]+\binom{0}{0}+7\left[\binom{3}{3}+\binom{3}{2}+\binom{3}{1}+\binom{3}{0}\right]+4\left[\binom{1}{1}+\binom{1}{0}\right]$$

$$=3\left[\left(\frac{2!}{2!*0!}\right)+\left(\frac{2!}{1!*1!}\right)+\left(\frac{2!}{0!*2!}\right)\right]+\left(\frac{0!}{0!*0!}\right)+7\left[\left(\frac{3!}{3!*0!}\right)+\left(\frac{3!}{2!*1!}\right)+\left(\frac{3!}{1!*2!}\right)+\left(\frac{3!}{0!*3!}\right)\right]+4\left[\left(\frac{1!}{1!*0!}\right)+\left(\frac{1!}{0!*1!}\right)\right]$$

$$=3(4)+1+7(8)+4(2)=77 \text{ potential spatio-temporal combinations.}$$

The matrix of dbh values (cm) for each tree at each time is

$$\begin{array}{c} \text{Year} \quad 10 \quad 9 \quad 8 \quad 7 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad \text{Tree} \end{array}$$

$$_{5,10}\mathbf{D}=\begin{bmatrix} 29 & 28 & 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 \\ 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 & 0 & 0 \\ 0 & 21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 37 & 35 & 33 \\ 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}.$$

Assume also that the bole centers of trees A through E, respectively, within the 1,600 m^2 area had coordinates (in m) of (12, 12), (19, 15), (23, 32), (23, 20), and (30, 20). Figure A1 shows the three-dimensional population highlighting a panel 1 selection. From this figure, all sampling probabilities can be derived for this population. Figure A2 gives the simplest two-dimensional depiction of the probability space in this probability proportional to size sample design for the three-dimensional population. The map of the tree bole centers is in the upper left graph and the probability spaces for each outcome

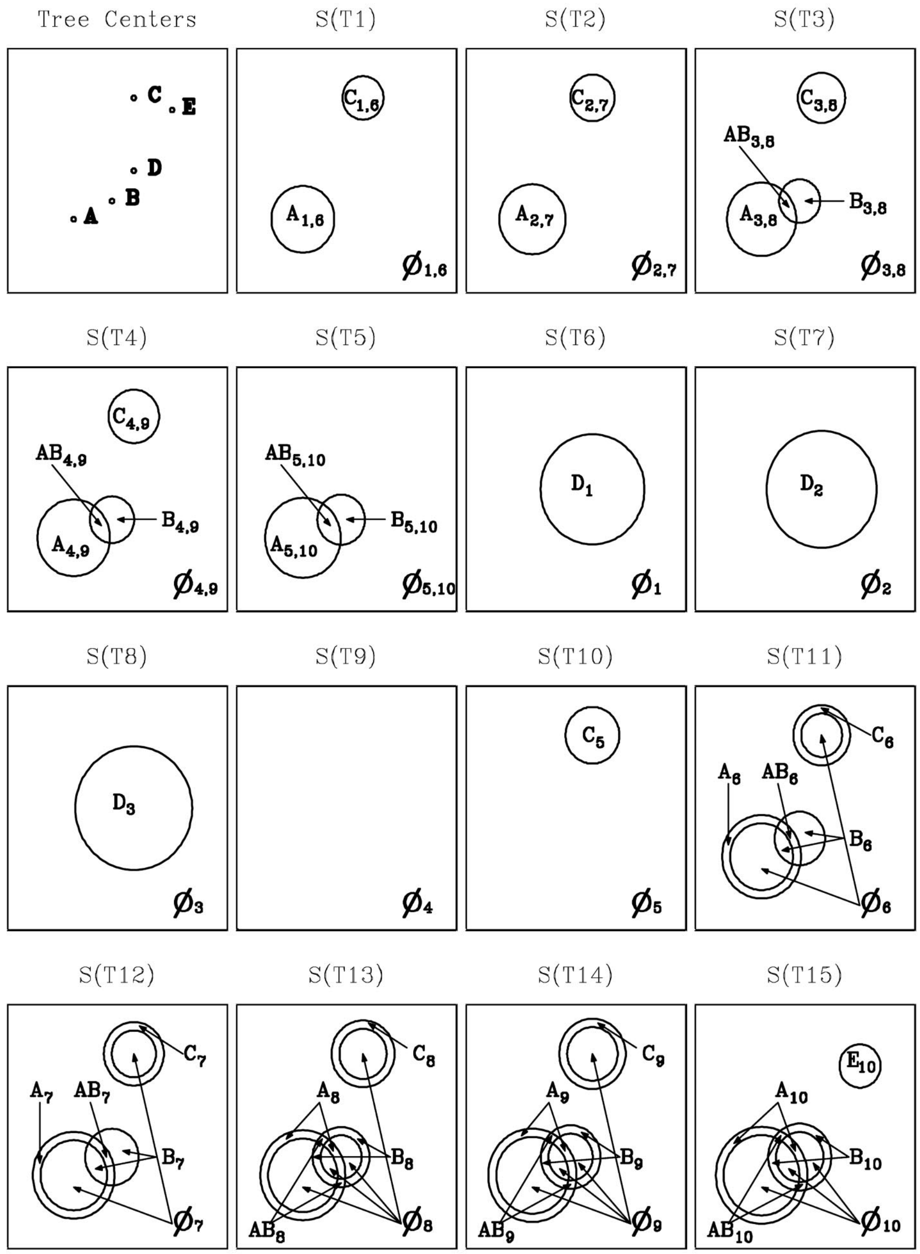

Figure A2. The tree centers (upper left) and K-circles for each set of outcomes for each of the 15 temporal partitions of the population in the Appendix.

for each of the 15 temporal partitions in the population appear in the succeeding 15 graphs. Incorporating knowledge of the spatial distribution, we see that the potential outcomes for each partition are reduced to

$$S(T_1) = \{A_{1,6}, C_{1,6}, \varnothing_{1,6}\} \qquad S(T_6) = \{D_1, \varnothing_1\} \qquad S(T_{11}) = \{A_6, B_6, C_6, AB_6, \varnothing_6\}$$

$$S(T_2) = \{A_{2,7}, C_{2,7}, \varnothing_{2,7}\} \qquad S(T_7) = \{D_2, \varnothing_2\} \qquad S(T_{12}) = \{A_7, B_7, C_7, AB_7, \varnothing_7\}$$

$$S(T_3) = \{A_{3,8}, B_{3,8}, C_{3,8}, AB_{3,8}, \varnothing_{3,8}\} \qquad S(T_8) = \{D_3, \varnothing_3\} \qquad S(T_{13}) = \{A_8, B_8, C_8, AB_8, \varnothing_8\}$$

$$S(T_4) = \{A_{4,9}, B_{4,9}, C_{4,9}, AB_{4,9}, \varnothing_{4,9}\} \qquad S(T_9) = \{\varnothing_4\} \qquad S(T_{14}) = \{A_9, B_9, C_9, AB_9, \varnothing_9\}$$

$$S(T_5) = \{A_{5,10}, B_{5,10}, AB_{5,10}, \varnothing_{5,10}\} \qquad S(T_{10}) = \{C_5, \varnothing_5\} \qquad S(T_{15}) = \{A_{10}, B_{10}, E_{10}, AB_{10}, \varnothing_{10}\}$$

This results in $M = 54$ spatial-temporal combinations. The entire three-dimensional sample frame consists of the union of the sets in $S(T_1)$ through $S(T_{15})$.