Application of the Algebraic Difference Approach for Developing Self-Referencing Specific Gravity and Biomass Equations

Lewis Jordan, Ray Souter, Bernard Parresol, and Richard F. Daniels

Abstract: Biomass estimation is critical for looking at ecosystem processes and as a measure of stand yield. The density-integral approach allows for coincident estimation of stem profile and biomass. The algebraic difference approach (ADA) permits the derivation of dynamic or nonstatic functions. In this study we applied the ADA to develop a self-referencing specific gravity function and biomass function as part of a density-integral system composed of taper, volume, specific gravity, and biomass functions. This was compared to base systems of similar equations that did not have the self-referencing parameter specifications. Systems of equations were fit using nonlinear, seemingly unrelated regressions with nonlinear cross-equation constraints to account for contemporaneous correlations in the data. Results suggest that correct volume determination is more critical than specific gravity for accurate biomass estimates. The goodness-of-fit statistics clearly show that the self-referencing system provided a better fit than the base system. FOR. SCI. 52(1):81–92.

Key Words: Algebraic difference approach, biomass, height invariant, seemingly unrelated regression, specific gravity.

BIOMASS AS AN ESTIMATION of wood quantity has been used extensively in northern Europe and the southeastern United States (Husch et al. 1982). While biomass has become increasingly important as a measure, so too is the increased interest in development of merchantable biomass equations. Merchantable biomass equations allow for prediction of biomass by product class from a tree, and hence the economic value of trees (Busby and Ward 1989). To achieve more efficient utilization of timber resources, forest managers need a thorough understanding of the relationships among volume, biomass, and wood properties, i.e., wood density (Myers et al. 1980). The laws of physics tell us that two objects with equal volumes and differing densities will have varying weights. Thus, trees with differing densities or specific gravity values, regardless of tree form and taper, will have varying total tree and merchantable biomass values.

In a data variability study examining the variation in the specific gravity of loblolly pine (*Pinus taeda* L.) increment cores from 30 randomly selected trees from each of 130 stands, Jordan et al. (2004) found that more variation in cross-sectional specific gravity was found to exist between stands than among trees within stands. This is an indicator that other factors, including site quality, length of growing season, rainfall, and genetics, could possibly be playing a key role in determining specific gravity. This also suggests that, if an estimate of tree or average stand specific gravity could be incorporated into density-integral type biomass equations, then more precise estimates of tree and subsequently stand biomass could be obtained. Not only would this allow for better estimates of standing biomass, but it would provide more precise estimates of the economic return expected.

The objective of this article is derivation of a dynamic system of compatible taper, volume, self-referencing specific gravity, and self-referencing biomass equations. The self-referencing specific gravity and subsequently derived biomass equations were developed using the algebraic difference approach method first proposed by Bailey and Clutter (1974), and discussed extensively by Cieszewski and Bailey (2000). This new approach will allow for the incorporation of observed specific gravity values in the biomass equation, which will lead to a more precise estimate of stem biomass.

Methodology

The Density-Integral Approach

The mass of an object is simply its volume multiplied by its density. This is well known, and can be found in any calculus book referencing the mass of a lamina. This fact is the basis for calculation of bolt dry mass from sample trees used in biomass studies. The normal procedure is to cut

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disks from the base of each bolt for laboratory determination of its density or its specific gravity. Since the base of a bolt is the top of the previous bolt, a measure of density at the ends of each bolt (except the tip) is obtained. A weighted average bolt density (where weight is disk cross-sectional area) is determined, which is then multiplied by bolt volume to obtain bolt dry mass. The need for weighting density results from the fact that it varies throughout the tree. Density, or specific gravity based on a green volume/dry weight basis, varies as a function of tree height, decreasing from the base to the tip of the stem (Haygreen and Bowyer 1996, Koch 1972, Phillips 2002). A taper equation explaining the change in tree diameter with height can be integrated between any two points along a stem to obtain sectional volume. Parresol and Thomas (1987, 1989) proposed a method for predicting the bole dry mass of wood by integrating the product of a taper and a density function. Applying some basic fundamentals of calculus, stem biomass can be defined as

$$B_{\mathrm{m}} = k\int_{h_{\mathrm{l}}}^{h_{\mathrm{u}}} \rho(h)d(h)^2\,\mathrm{d}h, \tag{1}$$

where B_{m} is merchantable biomass between the limits of h_{l} and h_{u}, $\rho(h)$ is a function expressing density as a function of height, $d(h)$ is a function expressing the taper in diameter as a function of height, h_{l} is the lower limit of integration, h_{u} is the upper limit of integration, and k is a constant for converting diameter to cross-sectional area.

Parresol and Thomas (1989) found that even a simple taper model combined with a specific gravity function resulted in superior fits of biomass when compared to the more popular biomass-ratio approach. The residuals of the fit were found to behave in a logical fashion, and error variance was considerably reduced. This is intuitively correct given the inherent association among mass, volume, and density. Empirically derived biomass equations have no physically meaningful mathematical relationships and assume stem diameter, or the ratio of diameters, is a function of breast height diameter, total height, height ratio, or some transformation of these variables (Fang and Bailey 1999). In addition, it is often difficult to explain the coefficients of empirical models, and the addition of multiple regressor variables may lead to multicollinearity and over-parameterization.

The Algebraic Difference Approach Method

Bailey and Clutter (1974) introduced the concept of deriving base-age invariant site equations using a technique now known as the algebraic difference approach (ADA) method. The ADA generally consists of replacing an arbitrary base model parameter with its initial condition solution. Initial condition difference equations are considered a part of differential calculus, and can be viewed as the procedure for boundary solutions in differential equations (Cieszewski 2001). The ADA approach uses repeated measures, or longitudinal data, collected over time from permanent plots or individuals to capture unobservable influences on the entity being observed. The ADA technique allows for the derivation of dynamic or nonstatic functions that are capable of producing anamorphic or polymorphic curves.

Following the theory presented by Cieszewski and Bailey (2000), the base function Y as a function of t and n individual specific parameters $\mathrm{P}_1 \cdots \mathrm{P}_{n-1}$ may be written as

$$Y(t) = f(t, \mathrm{P}_1 \cdots \mathrm{P}_{n-1}, \mathrm{P}_n). \tag{2}$$

The solution for any arbitrary parameter P_n is a function of two independent variables Y and t, and the $n - 1$ remaining parameters. Y and t in the solution are independent variables and can be assigned the values of Y_0 and t_0, resulting in

$$\mathrm{P}_n = u(t, Y, \mathrm{P}_1 \cdots \mathrm{P}_{n-1}) = u(t_0, Y_0, \mathrm{P}_1 \cdots \mathrm{P}_{n-1}), \tag{3}$$

where Y_0 is a given value of Y for an arbitrary t_0.

The solution in Equation 3 can be used in place of P_n in the base function to define a new dynamic function of time t, an arbitrary time t_0, a given function value Y_0 at t_0, and the remaining $n - 1$ parameters given as

$$Y(t, t_0, Y_0) = w(t, t_0, Y_0, \mathrm{P}_1 \cdots \mathrm{P}_{n-1}). \tag{4}$$

Equation 4 is undefined without the arbitrary initial conditional values of t_0 and Y_0. With t_0 and Y_0 assuming any value, the equation represents a dynamic equation that produces a curve unchanging under all choices of t_0. The model formulation also ensures that $Y_0 = Y$ at $t_0 = t$. The main limitation of the ADA is that all models derived with it are limited to the solution of only one individual specific parameter (Cieszewski 2001). This allows for the development of only anamorphic or polymorphic models.

Data

The data consisted of 70 felled plantation-grown slash pine trees. All trees were cut at a 0.15-m stump. Diameter inside bark (cm) was measured at 0.15, 0.6, and 1.4 m and every 1.5 m thereafter throughout the remainder of the stem. Total tree height (m) was measured, and the stem was sectioned into bolts. After each bolt was weighed, a 4-cm-thick disk was cut off the bottom end for laboratory determination of wood specific gravity on a dry weight/green volume basis. Green bolt volumes inside bark were calculated using Smalian's formula. A weighted (by cross-sectional area) average wood specific gravity was computed based on disks from the upper and lower end of each bolt, so that dry mass of the bolt (in kg) could be determined. The greatest specific gravities were found in the oldest trees in the sample. Trees with strikingly low taper coefficients obviously can be identified also. For these reasons, potential users should be doubly careful of application of the results to trees outside the reported ranges. The dbh (cm) ranged from 13.2 to 33.0 and averaged 19.5. Total height (m) ranged from 8.5 to 27.7 and averaged 17.6. Age (years) of the sample data ranged from 12 to 45 and averaged 25.8. A total of 902 observations were used in this study.

Analysis

Seemingly Unrelated Regression

Parresol and Thomas (1996) and Thomas et al. (1995) showed that a density-integral system of equations contains statistical dependencies and thus is best fit using nonlinear joint-generalized least-squares regression, also known as nonlinear seemingly unrelated regression (NSUR). A set of nonlinear regression functions are specified such that (1) each equation can contain its own independent variables, (2) each equation can use its own weight function (if needed), and (3) conformity (e.g., a volume function derived from a taper function) is ensured by setting constraints on the regression coefficients (i.e., parameter sharing). The structural equations for the system of nonlinear models can be specified as

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{f}_1(\mathbf{X}_1, \boldsymbol{\beta}_1) + \boldsymbol{\varepsilon}_1 \\ &\vdots \\ \mathbf{y}_2 &= \mathbf{f}_2(\mathbf{X}_2, \boldsymbol{\beta}_2) + \boldsymbol{\varepsilon}_2 \\ \mathbf{y}_k &= \mathbf{f}_k(\mathbf{X}_k, \boldsymbol{\beta}_k) + \boldsymbol{\varepsilon}_k, \end{aligned} \tag{5}$$

where $\mathbf{y}_i$ is a vector containing the dependent variable from the ith equation, $\mathbf{X}_i$ is a matrix containing the independent variables from the ith equation, $\boldsymbol{\beta}_i$ is the parameter vector for the ith equation, and $\boldsymbol{\epsilon}_i$ is the random error vector for the ith equation. When the stochastic properties of the error vectors are specified, along with the coefficient restrictions, the structural equations become a statistical model for efficient parameter estimates and reliable prediction intervals. The use of NSUR offers the best unbiased estimator for $\boldsymbol{\beta}_i$, which has a lower variance than the estimator of the ordinary nonlinear least-squares estimator of $\boldsymbol{\beta}_i$ because it takes into account contemporaneous correlation in the different equations. The availability of econometric software, such as SAS/ETS (SAS Institute Inc. Online Doc. http://v8doc.sas.com/sashtml/1999, accessed Summer 2005) makes complicated statistical procedures like NSUR easily implemented. It would be unrealistic to expect that the equation errors would be uncorrelated (Borders 1989, Parresol 1999, 2001). In an attempt to simultaneously minimize the error associated with these equations, the models in this article were fit as NSUR using the SAS/ETS model procedure.

Estimates of the self-referencing functions for each plot/individual may be obtained using the dummy variable method of Cieszewski et al. (1999). This method involves estimating the model's global parameters and the plot/individual parameters simultaneously. The dummy variable method recognizes that the measurements made at the referenced point were made with error. This method does not force the model through the specified point regardless of how the curve is fitted to the observed individual trends in the data. In this study we arbitrarily choose a base height, which thus becomes the reference point.

Comparison Criteria

The comparison of the models was based on graphical analysis of the residuals and four statistical indices: coefficient of determination (R^2) {In the forestry literature starting with Schlaegel (1982), one often sees the term "fit index" for Equation 6 applied to nonlinear models, but in the statistical literature it is still commonly referred to as the coefficient of determination, or R^2 [see, e.g., Kvålseth (1985)]}; root mean square error (RMSE); mean bias (MB); and mean absolute bias (MAB) (Loague and Green 1991, Mayer and Butler 1993). These criteria are given below as

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}, \tag{6}$$

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{n-p}}, \tag{7}$$

$$\text{MB} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_i), \tag{8}$$

$$\text{MAB} = \frac{1}{n}\sum_{i=1}^{n}|Y_i - \hat{Y}_i|, \tag{9}$$

where Y_i, $\hat{Y}$, and $\bar{Y}$ are the actual, predicted, and average values of the dependent variable; n is the total number of observations used to fit the model, and p is the number of parameters in the model. The models evaluated in this article were also examined visually by plotting the residuals against the estimated values and independent variables. Plots of the fitted curves for the self-referencing specific gravity and biomass equations were also examined given differing initial cross-sectional specific gravity values.

To truly assess performance of the models fit in this study, validation of the models with an independent data set would be the most desirable approach because quality of fit does not always ensure the quality of prediction (Huang et al. 2003, Kozak and Kozak 2003). The commonly used validation methods of data splitting and cross-validation, as shown by Kozak and Kozak (2003), do not provide any additional information on model performance compared to the statistics obtained from models fit to the entire data set. Models validated with an independent data set prove that either the data are from the same population and will perform as per se validation utilizing data splitting, or the data are from a different population entirely, in which case the models should be refit to obtain more appropriate parameter estimates. Due to the scarcity of data resembling that used herein, the models fit in this study were evaluated using the goodness-of-fit statistics and graphical analyses described above.

Model Development

Because this is an exercise to determine whether self-referencing implicitly defined specific gravity and biomass models are superior to the explicitly defined base equations, we have selected relatively simple mathematical models to focus more on the methodology for estimation in self-referencing systems.

A taper equation developed by Brister et al. (1980), which describes the change of diameter inside bark with height, was chosen as the base taper function for this study. The model has the form

$$\text{dib} = \beta_1 \text{dbh}^{\beta_2} H^{\beta_3} (H - h)^{\beta_4} + \varepsilon, \qquad (10)$$

where dib is diameter inside bark, dbh is diameter at breast height, H is total height, h is merchantable height of interest, and ϵ is residual error.

Volume inside the bark of the stem from the stump to some upper height limit can be found by integrating the taper equation, which is equal to

$$V_{\text{m}} = k \int_{h_{\text{l}}}^{h_{\text{u}}} \text{dib}(h)^2 \, \text{d}h, \qquad (11)$$

where V_{m} is merchantable volume inside bark from the stump to some upper height limit, $\text{dib}(h)^2$ is the taper function, $k = \pi/40{,}000$ in conventional metric units for converting dib^2 from cm^2 to area in m^2, h_{l} is the lower limit of integration (stump height), and h_{u} is the upper limit of integration. Integration of Equation 11 yields

$$V_{\text{m}} = -\left[\frac{k\beta_1{}^2 \text{dbh}^{2\beta_2} H^{2\beta_3}}{2\beta_4 + 1}\right] \times \left[(H - h_{\text{u}})^{2\beta_4+1} - (H - h_{\text{l}})^{2\beta_4+1}\right] + \varepsilon. \qquad (12)$$

Numerous specific gravity functions have been developed for the prediction of whole disk cross-sectional specific gravity at varying heights (Parresol and Thomas 1987, 1989, Parresol 1999, Phillips 2002, Zhang et al. 2002). These functions include the use of stem height, or some other height ratio transformation. Often, additional covariates are included in these models that may help to explain the variation of specific gravity with height. Parresol and Thomas (1987, 1989, 1996) and Thomas et al. (1995) proposed both linear and nonlinear models for estimating density as functions of relative height and age.

Zhang et al. (2002) found that density modeled as a quadratic function of height was a better fit for 12-year-old loblolly pine. Phillips (2002) describes several linear and nonlinear models for predicting cross-sectional specific gravity of loblolly pine in Georgia. Phillips also fit several self-referencing specific gravity functions that were derived from these nonlinear equations, finding that the self-referencing equations consistently provided better fits based on several statistical indices.

Several linear and nonlinear models were found to fit our data set well. On the basis of fit statistics and amenability to integration with the taper function Equation 10, we choose the model

$$\text{SG}_h = \phi_1 e^{-\phi_2 h/H} + \varepsilon. \qquad (13)$$

A plot of specific gravity versus disk height for all trees (not shown) indicated that an anamorphic model may best suit the data. We then fit Equation 13 allowing ϕ_1 then ϕ_2 to be tree-specific parameters. Fit statistics indicated that referencing the ϕ_1 parameter resulted in a better fit, confirming the need for anamorphic curves. Following Cieszewski and Bailey (2000), solving Equation 13 for ϕ_1 and setting the reference point at dbh produces the following height invariant self-referencing specific gravity equation:

$$\text{SG}_h = \left(\frac{\text{SG}_D}{e^{(-\phi_2 1.37/H)}}\right) e^{(-\phi_2 h/H)} + \varepsilon \qquad (14)$$

To ensure that the stochastic model was performing favorably, we fit the base equation (13) and the self-referencing equation (14) for comparative purposes. Parameter estimates for both equations are given in Table 1. From Table 1, it can be seen that the rate parameters of the two models are nearly equivalent. However, the standard error of the rate parameter in the self-referencing function is on the order of 50% lower than the base equation. R^2 and RMSE values for Equation 13 were 0.56 and 0.0410, and for Equation 14 were 0.89 and 0.0218. It can be seen that the unbiased parameter estimation of Equation 14 significantly improved model performance.

A plot of the self-referencing specific gravity equation (14) with varying initial dbh cross-sectional specific gravity values is shown in Figure 1. The curves in Figure 1 are anamorphic and predict higher specific gravity values up the tree stem given a larger initial value at dbh. Plots of the residuals (Figure 2) versus relative height and the predicted values indicate Equation 14 provides a better fit. The residual values from Equation 14 are more tightly centered on zero and show no apparent trends.

The mass equation can be defined as the integration of the product of the density and taper equations and is given as

$$B_{\text{m}} = k_1 k_2 \int_{h_{\text{l}}}^{h_{\text{u}}} \rho(h) \text{dib}(h)^2 \, \text{d}h, \qquad (15)$$

where B_{m} is merchantable biomass inside bark from the stump (h_{l}) to some upper height limit h_{u}, $\rho(h)$ is the specific gravity function, $\text{dib}(h)$ is the taper function, $k_1 = \pi/40{,}000$ in conventional metric units for converting dib^2 from cm^2 to area in m^2, and $k_2 = 1{,}000$ in conventional metric units for converting specific gravity to density (kg/m^3). Integration of

Table 1. Parameter estimates, standard errors, and *P* values for the base specific gravity equation (13) and the stochastic self-referencing equation (14)

Model	Parameter	Estimate	Standard error	*P* value
Eq. 13	ϕ_1	0.5352	0.0025	<0.0001
	ϕ_2	0.3365	0.0100	<0.0001
Eq. 14	ϕ_2	0.3387	0.0053	<0.0001

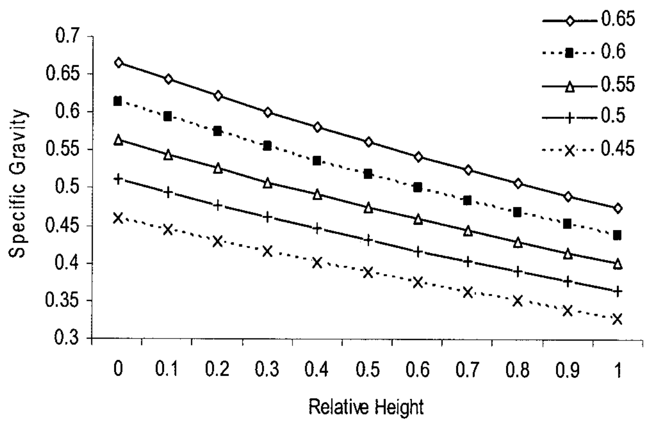

Figure 1. Curves produced by the self-referencing specific gravity equation (14) at a reference height of 1.37 meters with differing cross-sectional dbh initial specific gravity values.

Equation 15 using the self-referencing Equation 14 and base Equation 13 specific gravity models yields

$$B_{\mathrm{mSR}} = k_1 k_2 \int_{h_l}^{h_u} \left[\frac{\mathrm{SG}_D e^{(-\phi_2 h/H)}}{e^{(-\phi_2 1.37/H)}} \right] \times [\beta_1^2 \mathrm{dbh}^{2\beta_2} H^{2\beta_3} (H-h)^{2\beta_4}]\, \mathrm{d}h,$$

$$B_{\mathrm{m}} = k_1 k_2 \int_{h_l}^{h_u} [\phi_1 e^{(-\phi_2 h/H)}] \times [\beta_1^2 \mathrm{dbh}^{2\beta_2} H^{2\beta_3} (H-h)^{2\beta_4}]\, \mathrm{d}h, \quad (16)$$

where B_{mSR} and B_{m} denote biomass of the self-referencing and base equations, respectively. Extracting the constants and simplifying gives

$$B_{\mathrm{mSR}} = k_1 k_2 \beta_1^2 \mathrm{dbh}^{2\beta_2} H^{2\beta_3} \frac{\mathrm{SG}_D}{e^{(-\phi_2 1.37/H)}} \times \int_{h_l}^{h_u} (H-h)^{2\beta_4} e^{(-\phi_2 h/H)}\, \mathrm{d}h,$$

$$B_{\mathrm{m}} = k_1 k_2 \beta_1^2 \mathrm{dbh}^{2\beta_2} H^{2\beta_3} \phi_1 \times \int_{h_l}^{hu} (H-h)^{2\beta_4} e^{(\phi_2 h/H)}\, \mathrm{d}h. \quad (17)$$

Integration of Equation 17 for the self-referencing function yields the merchantable dry biomass of the tree stem from the stump (h_l) to some merchantable height limit (h_u) and is given as

$$B_{\mathrm{mSR}} = \left[k_1 k_2 \beta_1^2\, \mathrm{dbh}^{2\beta_2}\, H^{2\beta_3} \frac{\mathrm{SG}_D(-e^{-\phi_2}H)}{e^{(-\phi_2 1.37/H)}\phi_2} \right] \times \left[\begin{array}{l} (H-h_u)^{2\beta_4}\left(-\dfrac{\phi_2(H-h_u)}{H}\right)^{-2\beta_4} \Gamma\left(1+2\beta_4, -\dfrac{\phi_2(H-h_u)}{H}\right) \\ -(H-h_l)^{2\beta_4}\left(-\dfrac{\phi_2(H-h_l)}{H}\right)^{-2\beta_4} \Gamma\left(1+2\beta_4, -\dfrac{\phi_2(H-h_l)}{H}\right) \end{array} \right] + \varepsilon. \quad (18)$$

The merchantable dry biomass equation using the base specific gravity model follows that of the self-referencing function and is given as

$$B_{\mathrm{m}} = \left[k_1 k_2 \beta_1^2\, \mathrm{dbh}^{2\beta_2}\, H^{2\beta_3}\, \phi_1 \frac{(-e^{-\phi_2}H)}{\phi_2} \right] \times \left[\begin{array}{l} (H-h_u)^{2\beta_4}\left(-\dfrac{\phi_2(H-h_u)}{H}\right)^{-2\beta_4} \Gamma\left(1+2\beta_4, -\dfrac{\phi_2(H-h_u)}{H}\right) \\ -(H-h_l)^{2\beta_4}\left(-\dfrac{\phi_2(H-h_l)}{H}\right)^{-2\beta_4} \Gamma\left(1+2\beta_4, -\dfrac{\phi_2(H-h_l)}{h}\right) \end{array} \right] + \varepsilon. \quad (19)$$

The term

$$\Gamma\left(1+2\beta_4, -\frac{\phi_2(H-h_x)}{H}\right)$$

resulting from the integration of Equation 17 is an incomplete gamma function. It should be noted that, even though a closed form solution exists for the integrals presented in Equation 17, evaluation of

$$\Gamma\left(1+2\beta_4, -\frac{\phi_2(H-h_x)}{H}\right)$$

will result in a complex argument, i.e., an imaginary number. This arises from the negative coefficient in the second argument of the incomplete gamma function and takes one into the realm of complex analysis. Several methods are available for evaluating this complex function, including a hypergeometric power series, Laurent series, or power series of the half argument, and numerical integration (Mathar 2004).

We chose the two-point closed Newton–Cotes formula called the "trapezoid rule" for numerical integration. We then wrote code using the SAS/MACRO function to evaluate only the integral presented in Equation 17 using the trapezoid rule, because the terms outside of the integrand are constants and parameters to be estimated. We then implemented this macro within the SAS/ETS Proc Model procedure, which evaluates the integral and simultaneously estimates the parameters. An iterative procedure is used, with a criterion for convergence being agreement of the estimated integral between successive iterations to within 1×10^{-6}. Interested readers can contact the corresponding author for the SAS code used in this article. As pointed out by one anonymous reviewer, evaluating the incomplete gamma function of negative arguments in Equations 18 and 19 requires more advanced and difficult numerical analysis techniques than the numerical integration method we chose. However, Equations 18 and 19 can still be used in conjunction with one of the methods described above for evaluating complex functions. Thus, when referring to Equations 18 and 19 throughout the study, we are indirectly referring to Equation 17. A proof of the integral presented in Equation 17 is found in the Appendix.

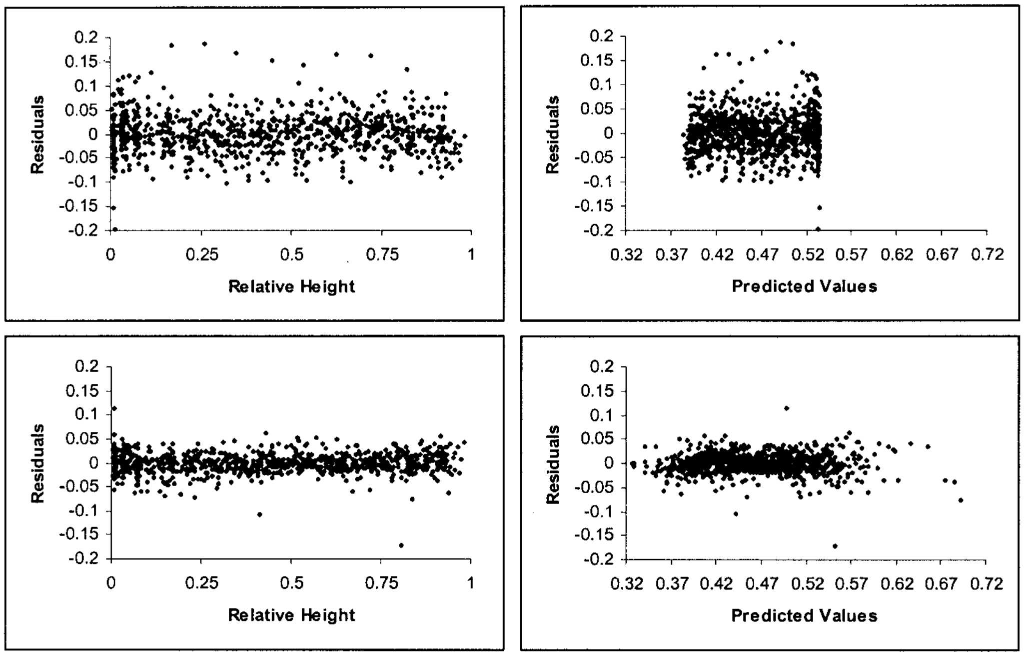

Figure 2. First row, residuals versus relative height and predicted values for the base specific gravity equation (13). Second row, the stochastic self-referencing equation (14).

Results

Initial Results

We fit the system of self-referencing specific gravity and merchantable dry biomass equations, along with the taper and volume equations denoted as SYSSR (Equations 10, 12, 14, 18) for comparison against the base system of equations SYSBASE (Equations 10, 12, 13, 19). Parameter estimates for the respective model fits are presented in Table 2. All parameter estimates for both system fits were found to be significant at the 0.0001 level, are logical, and ensure compatibility between equations. The specific gravity rate parameter ϕ_2 was found to vary significantly between the SYSBASE and SYSSR fit. The parameter values were found to be 0.3326 and 0.2676 for the SYSBASE and SYSSR systems, respectively. The rate parameter for the self-referencing model from the system fit is also substantially different from the value found when Equation 14 was fit independently (Table 1).

Fit statistics for the system fits are presented in Table 3. It can be seen that the self-referencing mass equation (18) outperforms the base mass equation (19), which is evident by the fit statistics presented. It can also be seen that both specific gravity functions possess similar fit statistics. However, the self-referencing specific gravity function R^2 has decreased significantly compared to the independent fit of Equation 14. Because the self-referencing biomass equation (18) shares common parameters with both the volume and self-referencing specific gravity functions, this suggests that the system fit is disregarding the specific gravity equation and fitting parameters that best estimate volume and biomass. This indicates that precise estimates of volume are more important in predicting stem biomass than is stem-specific gravity. On examination of the cross-equation correlation matrix (Table 4), it was found that the volume equation (12) and the self-referencing specific gravity equation (14) were highly negatively correlated. This suggests that, if volume is overestimated, specific gravity will have to be grossly underestimated to predict the appropriate mass. Similarly, if volume is underestimated, specific gravity will be overestimated to obtain a more precise prediction of stem mass. Plots of the residual specific gravity values

Table 2. Parameter estimates, standard errors, and *P* values for the SYSBASE (Equations 10, 12, 13, 19) and SYSSR (Equations 10, 12, 14, 18) system fits

	SYSBASE			SYSSR		
Parameter	Estimate	SE	*P* value	Estimate	SE	*P* value
β_1	0.6116	0.0114	0.0001	0.7264	0.0109	0.0001
β_2	0.9770	0.0085	0.0001	1.0376	0.0071	0.0001
β_3	−0.4446	0.0107	0.0001	−0.53159	0.0083	0.0001
β_4	0.5971	0.0059	0.0001	0.5612	0.0034	0.0001
ϕ_1	0.5381	0.0021	0.0001			
ϕ_2	0.3326	0.0111	0.0001	0.2676	0.0105	0.0001

Table 3. Fit statistics of the SYSBASE (Equations 10, 12, 13, 19) and SYSSR (Equations 10, 12, 14, 18) system fits, where R^2 is the coefficient of determination, RMSE is the root-mean-square error, MB is the mean bias, and MAB is the absolute mean bias

	R^2	RMSE	MB	MAB
SYSBASE				
Eq. 10	0.9604	1.1378	0.1408	0.8132
Eq. 12	0.9891	0.0158	−0.0015	0.0098
Eq. 13	0.5616	0.0411	−0.0032	0.0303
Eq. 19	0.9891	7.9284	−0.6939	4.543
SYSSR				
Eq. 10	0.9596	1.1494	−0.0463	0.8066
Eq. 12	0.9905	0.0147	−0.0022	0.0088
Eq. 14	0.5702	0.0415	0.0032	0.0323
Eq. 18	0.9996	1.4607	0.1404	0.9497

versus height (not shown here) indicated that individual trees were constantly being either over or underpredicted for specific gravity.

Additional Model Development

On finding that the self-referencing specific gravity equation (14) parameter estimates were being disregarded to fit the best self-referencing biomass and volume models, we then fit a similar system of equations estimating mass using the volume as calculated by Smalian's formula and the self-referencing specific gravity function. This will ensure that the correct volume is being used in the mass equation, thus allowing for the best estimates of the self-referencing specific gravity and biomass equations. Smalian's taper equation is defined as

$$\mathrm{dib}^2 = D_1^2 + \frac{(h - H_1)(D_2^2 - D_1^2)}{H_2 - H_1}, \tag{20}$$

where D_1 is diameter inside bark at the base of the bolt, D_2 is diameter inside bark at the top of the bolt, H_1 is height at the base of the bolt, H_2 is height at the top of the bolt, and h is height of interest. We want to be able to calculate the volume between any two points along the tree stem. Thus,

Table 4. Cross-equation correlation matrix of residuals for the SYSBASE (Equations 10, 12, 13, 19) and SYSSR (Equations 10, 12, 14, 18) system fits

SYSBASE	Eq. 10	Eq. 12	Eq. 13	Eq. 19
Eq. 10	1	0.2993	−0.2037	0.2151
Eq. 12		1	−0.2384	0.5529
Eq. 13			1	0.2828
Eq. 19				1
SYSSR	Eq. 10	Eq. 12	Eq. 14	Eq. 18
Eq. 10	1	0.3522	−0.3112	−0.0507
Eq. 12		1	−0.4832	0.2990
Eq. 14			1	0.2126
Eq. 18				1

we integrate Equation 20 from H_1 to H_2, yielding Smalian's volume equation,

$$V = k\int_{H_1}^{H_2} D_1^2 + \frac{(h - H_1)(D_2^2 - D_1^2)}{H_2 - H_1}\,\mathrm{d}h$$

$$= k\,\frac{(D_1^2 + D_2^2)(H_2 - H_1)}{2}, \tag{21}$$

where k is the conversion constant $\pi/40{,}000$ for converting diameter inside bark into cross-sectional area.

The mass of the stem between the limits of H_1 and H_2 can be found by integrating the product of the self-referencing specific gravity equation (14) and Smalian's taper formula, which gives

$$B_{\mathrm{mSR}} = k\int_{H_1}^{H_2}\left[D_1^2 + \frac{(h - H_1)(D_2^2 - D_1^2)}{H_2 - H_1}\right]\left[\frac{SG_D}{e^{-\phi_2(1.37/H)}}\,e^{-\phi_2(h/H)}\right]\mathrm{d}h$$

$$= \frac{kSG_DHe^{(-\phi_2(-1.37+H_2+H_1)/H)}}{\phi_2^2(-H_2+H_1)}$$

$$\times\left[\begin{array}{l} e^{(\phi_2H_1/H)}D_2^2H_2\phi_2 + e^{(\phi_2H_1/H)}D_2^2H - e^{(\phi_2H_1/H)}D_1^2H \\ \quad - e^{(\phi_2H_1/H)}D_2^2H_1\phi_2 - e^{(\phi_2H_2/H)}D_1^2H_2\phi_2 + e^{(\phi_2H_2/H)}D_1^2H_1\phi_2 \\ \quad - e^{(\phi_2H_2/H)}D_2^2H + e^{(\phi_2H_2/H)}D_1^2H \end{array}\right] + \varepsilon \tag{22}$$

Similarly, stem biomass estimated as a function of Smalian's taper and the base specific gravity equation is given as

$$B_{\mathrm{m}} = \frac{kH\phi_1e^{(\phi_2(H_2+H_1)/H)}}{\phi_2^2(-H_2+H_1)}$$

$$\times\left[\begin{array}{l} e^{(\phi_2H_1/H)}D_2^2H_2\phi_2 + e^{(\phi_2H_1/H)}D_2^2H - e^{(\phi_2H_1/H)}D_1^2H \\ \quad - e^{(\phi_2H_1/H)}D_2^2H_1\phi_2 - e^{(\phi_2H_2/H)}D_1^2H_2\phi_2 + e^{(\phi_2H_2/H)}D_1^2H_1\phi_2 \\ \quad - e^{(\phi_2H_2/H)}D_2^2H + e^{(\phi_2H_2/H)}D_1^2H \end{array}\right] + \varepsilon. \tag{23}$$

Because we are interested in the cumulative mass of the tree stem from the stump to some upper merchantable height, we wrote an algorithm using SAS/BASE, which evaluates Equations 22 and 23, and adds the subsequently determined bolt mass estimates for a cumulative mass estimate. We then implemented this algorithm within the SAS/ETS model procedure. To ensure that this procedure was working correctly, we fit Equation 22 independently and then Equations 22 and 14 as a system fit. If the correct volume is being used in estimating stem biomass, then the parameter estimates for the self-referencing specific gravity equation (14) should provide the best and unbiased parameter estimates. The estimated values of the rate parameters should be approximately the same whether the equations are fitted independently or as a system.

Table 5. Parameter estimates, standard errors, P values, R^2, and root-mean-square error (RMSE) values for Equation 22 fit using Smalian's volume formula and the system fits of the stochastic self-referencing specific gravity and biomass equations (14 and 22)

Model	Parameter	Estimate	Standard error	P value	R^2	RMSE
Eq. 22	ϕ_2	0.3633	0.0035	0.0001	0.9999	0.6392
Equation	ϕ_2	0.3604	0.0030	0.0001		
22					0.9999	0.6333
14					0.8399	0.0253

Results of Additional Model Development

Table 5 contains the parameter estimates and fit statistics of the independent fit of Equation 22 and the system fit of Equations 22 and 14. It can be seen from Table 5 that the estimates of the rate parameter are approximately equivalent. The high R^2 values from both fits indicate that the volume algorithm is performing, and that Smalian's volume is being used in computing cumulative stem biomass.

The utility of Equations 22 and 23 is limited in the fact that Smalian's taper equation assumes a linear change in cross-sectional area and would only be applicable for estimating stem biomass between relatively short bolt lengths. However, computing cumulative stem biomass using the correct volume ensures that the best parameter estimates will be obtained for the self-referencing specific gravity function and the taper and volume functions, respectively. We then proceeded to simultaneously fit the taper and the volume equations with the respective base and self-referencing specific gravity and biomass equations, denoted as SYSBASE2 (Equations 10, 12, 13, 23), representing the base functions fit, and SYSSR2 (Equations 10, 12, 14, 22) the self-referencing system fit. Even though we are using the volume derived from Smalian's formula, we fit the original taper and volume equations to account for possible cross-equation correlation, and to enhance the utility of the system. Because application of the biomass equations derived using Smalian's volume formula are not practical, the parameter estimates obtained can be used in conjunction with Equation 17, or the derived self-referencing biomass equation (18), and should produce more precise estimates of stem biomass.

Parameter estimates for the system fits using the volume obtained from Smalian's volume equation are given in Table 6. The rate parameter for the self-referencing specific gravity function (Equation 14) was found to increase significantly from 0.2676 (Table 2) to 0.3615. The rate parameter for the self-referencing specific gravity and biomass equations is the parameter that best estimates specific gravity and biomass jointly, and is not influenced by volume. Fit statistics for the SYSBASE2 and SYSSR2 system fits are presented in Table 7. It can be seen from Table 7 that using the true volume improved the estimation of biomass for both the SYSBASE2 and SYSSR2 system fits. However, the RMSE of the base and biomass equations differed greatly and were found to be 7.41 and 0.63, respectively. The fit index values of the self-referencing specific gravity equation increased from 0.5702 to 0.8403, and the RMSE decreased from 0.0415 to 0.0253 when compared to the fit statistics presented in Table 3. As expected, the fit statistics for the original taper (Equation 10) and volume (Equation 12) equations are nearly identical for both the SYSBASE2 and SYSSR2 system fits. This should be the case, because neither model was used to estimate stem biomass, and thus neither share nor influence the parameters in the respective specific gravity and biomass equations.

Plots of the residuals versus relative height and the predicted values for the base (Equation 23) and self-referencing (Equation 22) biomass equations are shown in Figure 3. It can be seen that the residuals versus relative height for the self-referencing biomass function are more tightly centered on zero and indicate no trends, ranging from 4 to −3 kg. The residuals for the base equation versus relative height were found to vary substantially and ranged from 40 to −20 kg. The mean residual value for the base and self-referencing biomass equations were found to be −0.3208 and 0.0750, respectively. Plots of the residuals versus predicted biomass values for the base equation indicate patterns of heteroskedasticity, whereas the self-referencing biomass equation does not exhibit signs of nonconstant variance.

Table 6. Parameter estimates, standard errors, and P values for the SYSBASE2 (Equations 10, 12, 13, 23) and SYSSR2 (Equations 10, 12, 14, 22) system fits using Smalian's volume formula

	SYSBASE2			SYSSR2		
Parameter	Estimate	SE	P value	Estimate	SE	P value
β_1	0.6773	0.0131	0.0001	0.7083	0.0137	0.0001
β_2	1.0223	0.0090	0.0001	1.0522	0.0090	0.0001
β_3	−0.5266	0.0108	0.0001	−0.5741	0.0108	0.0001
β_4	0.5984	0.0056	0.0001	0.6007	0.0056	0.0001
ϕ_1	0.5438	0.0015	0.0001			
ϕ_2	0.3463	0.0079	0.0001	0.3615	0.0030	0.0001

Table 7. Fit statistics of the SYSBASE2 and SYSSR2 system fits using Smalian's volume formula, where R^2 is the coefficient of determination, RMSE is the root-mean-square error, MB is the mean bias, and MAB is the absolute mean bias

	R^2	RMSE	MB	MAB
SYSBASE2				
Eq. 10	0.9623	1.1103	0.1045	0.7951
Eq. 12	0.9904	0.0148	−0.0015	0.0089
Eq. 13	0.5551	0.0414	−0.0056	0.0308
Eq. 23	0.9904	7.4162	−0.3208	4.1463
SYSSR2				
Eq. 10	0.9628	1.1038	0.0773	0.7963
Eq. 12	0.9903	0.0149	−0.0019	0.0088
Eq. 14	0.8403	0.0253	−0.0114	0.0189
Eq. 22	0.9999	0.6346	0.0750	0.3865

A plot of cumulative stem biomass versus relative height for the self-referencing biomass equation (Equations 17 or 18) using the parameter estimates obtained from the SYSSR2 system fit is given in Figure 4. It can be seen in Figure 4 that a higher initial dbh specific gravity value results in an increase in cumulative stem biomass. For a tree 20 m in height and 25 cm dbh, a 20 kg increase in total tree biomass results from an increase in specific gravity of 0.05.

Discussion

The use of merchantable biomass equations is a critical component of forest characterization. These models allow industry to assess the product yields of trees to capture the highest value from raw wood. Biomass is most important in assessing chip furnish yields for the paper industry, and is often used by the solid products industry as a rough approximation of tree volume. In this study we derived a system of anamorphic and height-invariant self-referencing specific gravity and biomass equations using the algebraic difference approach method. These equations will allow users to input individual tree or aggregated stand-level specific gravity information into the models for more precise prediction of stem/stand biomass. It is easy to derive an estimate of tree specific gravity by obtaining increment cores at breast height or another randomly selected height, such as done by Van Deusen and Baldwin (1993) and discussed by Parresol (1999). By simple extension an unbiased estimate of stand specific gravity is obtained from a

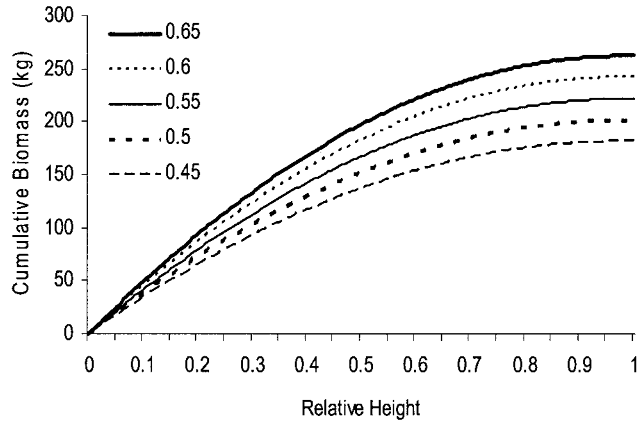

Figure 4. Curves produced by the self-referencing biomass equation (Equations 17 or 18) using the parameters obtained in the SYSSR2 system fit, at a reference height of 1.37 meters with differing dbh cross-sectional initial specific gravity values for a tree 20 m in height and 25 cm dbh.

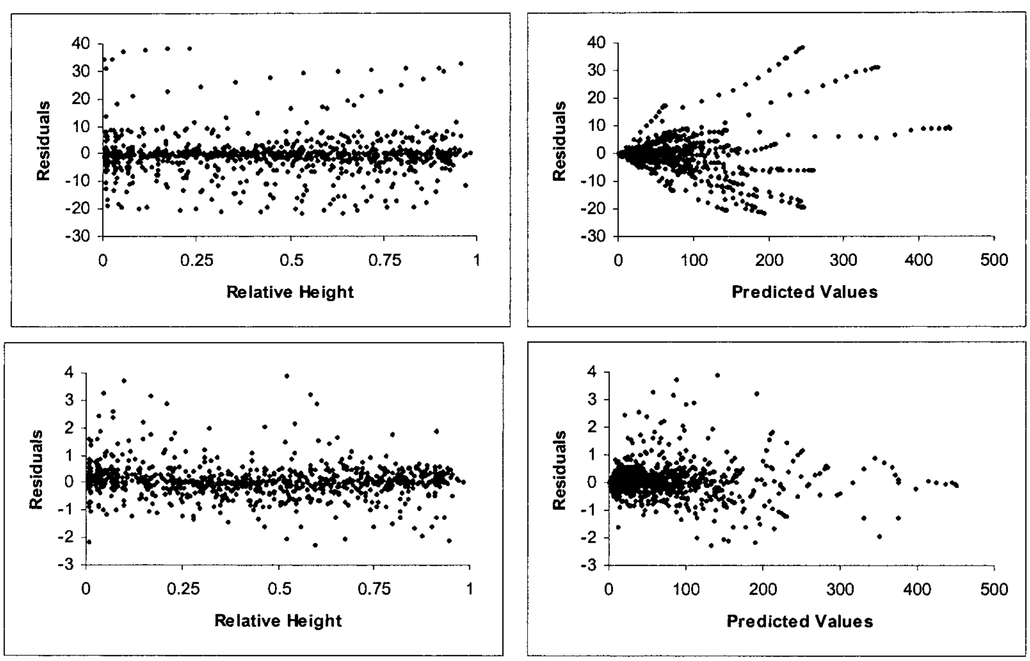

Figure 3. First row, residuals versus relative height and predicted values for the base biomass equation (23). Second row, the stochastic self-referencing biomass equation (22) using Smalian's volume formula.

weighted average of the specific gravity cores from a random sample of trees.

The self-referencing specific gravity and biomass equations presented in this article were found to be superior to the base equations, and resulted in more accurate estimates of stem specific gravity and biomass. This study indicates that precise estimates of taper, and subsequently volume, are more critical in accurately predicting biomass than specific gravity or density. It was found in this study that the parameter estimates of the self-referencing specific gravity equation (14) were being biased in the SYSSR fit to best fit the relationship between volume and biomass. We then fit a new system of equations (SYSSR2) using the volume calculated from Smalian's volume equation to estimate unbiased/uninfluenced parameters of the self-referencing specific gravity model. The SYSSR2 system fit provides the best parameter estimates for both taper and volume and the self-referencing specific gravity and biomass equations. These parameters can be used in conjunction with Equation 17, or the originally derived self-referencing biomass equation (18) for prediction of stem biomass to any merchantable height limit.

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Appendix

The indefinite integral

$$\int (H-x)^{2\beta_4} e^{(-\phi_2 x/H)}\,\mathrm{d}x$$

$$= e^{-\phi_2}(H-x)^{(1+2\beta_4)}\left(-\frac{\phi_2(H-x)}{H}\right)^{(-1-2\beta_4)} \times \Gamma\left(1+2\beta_4,\ -\frac{\phi_2(H-x)}{H}\right). \quad \text{(A1)}$$

It suffices to show that

$$(H-x)^{2\beta_4} e^{(-\phi_2 x/H)} = \frac{\mathrm{d}}{\mathrm{d}x} e^{-\phi_2}(H-x)^{(1+2\beta_4)}\left(-\frac{\phi_2(H-x)}{H}\right)^{(-1-2\beta_4)} \times \Gamma\left(1+2\beta_4,\ -\frac{\phi_2(H-x)}{H}\right). \quad \text{(A2)}$$

Using the product rule,

$$\frac{\mathrm{d}(u,v,w)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}cvw + \frac{\mathrm{d}v}{\mathrm{d}x}cuw + \frac{\mathrm{d}w}{\mathrm{d}x}cuv, \quad \text{(A3)}$$

where

$$c = e^{-\phi_2}, \quad u = (H-x)^{(+2\beta_4)},$$

$$v = \left(-\frac{\phi_2(\mathrm{H}-\mathrm{x})}{\mathrm{H}}\right)^{(-1-2\beta_4)},$$

$$\mathrm{w} = \Gamma\left(1+2\beta_4,\ -\frac{\phi_2(H-x)}{H}\right).$$

Using the chain rule where

$$\frac{\mathrm{d}u}{\mathrm{d}x}cvw = \frac{\mathrm{d}u_*^n}{\mathrm{d}x}cvw = nu_*^{n-1}\ \frac{\mathrm{d}u_*}{\mathrm{d}x}cvw$$

where $u_* = (H-x)$ and $n = (1+2\beta_4)$,

$$= [-(1+2\beta_4)(H-x)^{2\beta_4}]cvw. \quad \text{(A4)}$$

Using the chain rule

$$\frac{\mathrm{d}v}{\mathrm{d}x}cuw = \frac{\mathrm{d}vn_*^n}{\mathrm{d}x}cuw = nv_*^{n-1}\frac{\mathrm{d}v_*}{\mathrm{d}x}cuw$$

where $v^* = \left(-\frac{\phi(H-x)}{H}\right)$ and $n = (-1-2\beta_4)$.

$$= \left[\phi_2 H^{-1}(-1-2\beta_4)\left(\frac{-\phi_2(H-x)}{H}\right)^{(-2-2\beta_4)}\right]cuw. \quad \text{(A5)}$$

Let $w = \Gamma(a, z)$, where $a = (1+2\beta_4)$ and

$$z = \left(\frac{-\phi_2(H-x)}{H}\right),$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = -e^{-z}z^{a-1}\frac{\mathrm{d}z}{\mathrm{d}x}cuv$$

$$= -\phi_2 H^{-1}e^{(\phi_2(H-x)/H)}\left(-\frac{\phi_2(H-x)}{H}\right)^{2\beta_4}cuv. \quad \text{(A6)}$$

Combining Equations A4, A5, and A6, and substituting in c, u, v, and w from A3, gives

$$\frac{\mathrm{d}(u,v,w)}{\mathrm{d}x}$$

$$\begin{aligned} = &-(1+2\beta_4)(H-x)^{(2\beta_4)}e^{-\phi_2} \\ &\times\left(-\frac{\phi_2(H-x)}{H}\right)^{(-1-2\beta_4)} \\ &\times\Gamma\left(1+2\beta_4, -\frac{\phi_2(H-x)}{H}\right) \\ &+\phi_2 H^{-1}(-1-2\beta_4)\left(-\frac{\phi_2(H-x)}{H}\right)^{(-2-2\beta_4)} \\ &\times e^{-\phi_2}(H-x)^{(1+2\beta_4)} \\ &\times\Gamma\left(1+2\beta_4, -\frac{\phi_2(H-x)}{H}\right) \\ &-\phi_2 H^{-1}\left(-\frac{\phi_2(H-x)}{H}\right)^{(2\beta_4)}e^{(\phi_2(H-x)/H)} \\ &\times e^{-\phi_2}(H-x)^{(1+2\beta_4)}\left(-\frac{\phi_2(H-x)}{H}\right)^{(-1-2\beta_4)}. \end{aligned} \quad \text{(A7)}$$

Factoring the first two terms in Equation A7 and simplifying the third term gives

$$\frac{\mathrm{d}(u, v, w)}{\mathrm{d}x} = \Bigg[(-1 - 2\beta_4)(H - x)^{(2\beta_4)}e^{-\phi_2} \times \left(-\frac{\phi_2(H - x)}{H}\right)^{(-1-2\beta_4)} \times \Gamma\left(1 + 2\beta_4, -\frac{\phi_2(H - x)}{H}\right)\Bigg] \times \left[-1 - \phi_2 H^{-1}\left(-\frac{\phi_2(H - x)}{H}\right)(H - x)\right] - \frac{\phi_2 e^{(-\phi_2 x/H)}(H - x)^{(1+\beta_4)}}{\phi_2 Hx/H - \phi_2 H^2/H}. \quad \text{(A8)}$$

Further simplification of A8 yields

$$\frac{\mathrm{d}(u, v, w)}{\mathrm{d}x} = 0 - \frac{\phi_2 e^{(-\phi_2 x/H)}(H - x)^{(1+2\beta_4)}}{\phi_2 Hx/H - \phi_2 H^2/H} = -\frac{\phi_2 e^{(-\phi_2 x/H)}(H - x)^{(1+2\beta_4)}}{-\phi_2(H-x)} = (H - x)^{(2\beta_4)}e^{(-\phi_2 x/H)}. \quad \text{(A9)}$$