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Attractors for discrete periodic dynamical systems

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Abstract

A mathematical framework is introduced to study attractors of discrete, nonautonomous dynamical systems which depend periodically on time. A structure theorem for such attractors is established which says that the attractor of a time-periodic dynamical system is the union of attractors of appropriate autonomous maps. If the nonautonomous system is a perturbation of an autonomous map, properties that the nonautonomous attractor inherits from the autonomous attractor are discussed. Examples from population biology are presented.
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1. Introduction

This work develops a mathematical framework for studying attractors of discrete, nonautonomous dynamical systems which depend periodically on time. Continuous and discrete models of many physical and biological systems include periodic variation in both intrinsic and extrinsic parameters [1–3,5,8]. For models in population biology, periodicity in season and climate affects intrinsic parameters such as population growth rates, carrying capacities, and interaction coefficients and affects extrinsic factors such as stocking, harvesting and migration [6,7,11,13]. In addition, Güémez and Matías [4] illustrated how periodic stocking or harvesting may be used to produce stable periodic oscillation in pop-

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ulation size for logistic and exponential maps which behave chaotically without stocking or harvesting. Hence, chaos may be controlled in this setting by using periodic forcing.

The effects of periodic fluctuations have been observed in laboratory experiments. Jillson [8] studied the oscillations in population size of a flour beetle (*Tribolium*) given a periodic food supply. Henson and Cushing [7], Costantino et al. [1] and Henson et al. [6] explained Jillson's observations and suggested additional laboratory experiments by using a 3-dimensional, discrete model to study the flour beetle's behavior.

Motivated by models from population biology, here we study attractors for time-periodic, discrete dynamical systems. The periodicity permits us to consider an associated autonomous map on a topological cylinder constructed by including time as an additional state variable. The standard definitions for autonomous maps of concepts such as invariant sets and attractors may be used to define these concepts in the time-dependent setting. The compactness in the time direction avoids the technical details Thieme [14,15] needs to define attracting sets for nonautonomous semiflows. Using the cylinder space, we prove a structure theorem which states that an attractor of a time-periodic dynamical system is the union of attractors of appropriate autonomous maps. We establish conditions which guarantee these autonomous attractors are homeomorphic and the corresponding autonomous maps are conjugate. If the time-periodic system is a small C^1 perturbation of a diffeomorphism with a hyperbolic attractor then we show that these autonomous attractors are homeomorphic to each other and to the unperturbed hyperbolic attractor. Such perturbations arise from time-periodic forcing of an autonomous system, e.g., see Henson [5] or Selgrade and Roberds [13].

Section 2 presents examples of several attractors which occur in a time-dependent prey–predator model. These examples suggest that the topological structure of an attractor is related to the period of the time-variation and that its domain of attraction depends on time. Section 3 rigorously develops the mathematical framework for our study and proves the structure theorem for time-periodic attractors. Perturbations of autonomous maps and hyperbolicity are discussed in Sections 4 and 5.

2. An example of periodic variation in a prey–predator model

In this section we discuss a simple, 2-dimensional prey–predator system which experiences periodic variation in the intrinsic predator growth rate. Let x denote the prey population density and y, the predator density. We assume that the per capita transition functions are linear functions of the population densities and take parameter values so that the attractor in the positive quadrant is an invariant loop. In order to produce 2-periodic variation in an intrinsic parameter, we multiply the predator growth rate by the term $(1+\alpha(-1)^n)$ for $0 \leqslant \alpha < 1$. For $n = 1, 2, \ldots$, our system takes the form

$$\begin{aligned} x_n &= x_{n-1}(2 - x_{n-1} - 0.5y_{n-1}), \\ y_n &= y_{n-1}\big(0.8\big(1+\alpha(-1)^n\big) + 1.3x_{n-1}\big). \end{aligned} \tag{2.1}$$

We restrict our attention to small density values because if $y > 4$ or $x > 2$ then the prey density becomes negative in the next generation. If $\alpha = 0$, this system has an attracting

It is clear that an attractor Γ in $\mathcal{X}$ produces an attractor Λ in X for the time-periodic dynamical system.

To understand the structure of an attractor $\Lambda = \pi_X(\Gamma)$, consider the trapping region $\mathcal{U} \subset \mathcal{X}$ for Γ. This trapping region restricted to each fiber gives an open set which can be used to produce an attractor for an autonomous system in X. The autonomous system is the composition of all of the f_i taken in an appropriate order. The union of these attractors is Λ. The next theorem gives a precise statement of this observation and we will refer to it as the *Structure theorem*.

Theorem 3 (Structure theorem). *Let Λ be an attractor for the p-periodic dynamical system $\{f_0, f_1, \ldots, f_{p-1}\}$. Then $\Lambda = \bigcup_{i=0}^{p-1} \Lambda_i$, where Λ_i is an attractor for the map $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i : X \to X$, for $i = 0, 1, \ldots, p-1$.*

Proof. Assume Λ is an attractor for the p-periodic dynamical system. Then there is an attractor Γ for the cylinder map $\mathcal{F}$ such that $\pi_X(\Gamma) = \Lambda$. Since $\mathcal{F}(X_i) \subset X_{i+1 \bmod p}$ then $\mathcal{F}^p(X_i) \subset X_i$ for each fiber X_i. Now $\mathcal{F}(\Gamma) = \Gamma$, so $\mathcal{F}^p(\Gamma) = \Gamma$. Let $\Lambda_i = \pi_X(\Gamma \cap X_i)$, i.e., Λ_i is the projection of the part of Γ in the ith fiber. Thus $\Lambda = \bigcup_{i=0}^{p-1} \Lambda_i$ and $\mathcal{F}^p(\{i\} \times \Lambda_i) = \{i\} \times \Lambda_i$. Now $\mathcal{F}^p((i, x)) = (i, f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i(x))$. So $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i(\Lambda_i) = \Lambda_i$.

To see that Λ_i is an attractor for $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i$ we need to find a trapping region U_i. Since Γ is an attractor for $\mathcal{F}$, it has a trapping region $\mathcal{U}$. Now $\mathcal{F}(\bar{\mathcal{U}}) \subset \mathcal{U}$ so $\mathcal{F}^p(\bar{\mathcal{U}}) \subset \mathcal{U}$. Let $U_i = \pi_X(\mathcal{U} \cap X_i)$. Since $\mathcal{X}$ is a finite number of copies of X, $\overline{\mathcal{U} \cap X_i} = \bar{\mathcal{U}} \cap X_i$ and $\bar{U}_i = \pi_X(\bar{\mathcal{U}} \cap X_i)$. Using that $\mathcal{F}^p(\bar{\mathcal{U}} \cap X_i) \subset \mathcal{U} \cap X_i$ we see that $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i(\bar{U}_i) \subset U_i$. The final step is to get $\Lambda_i = \bigcap_{n=0}^{\infty} (f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i)^n(\bar{U}_i)$. To see this first note that $\mathcal{U} \supset \mathcal{F}(\bar{\mathcal{U}}) \supset \mathcal{F}^2(\bar{\mathcal{U}}) \supset \cdots \supset \mathcal{F}^p(\bar{\mathcal{U}}) \supset \cdots \supset$ implies that $\Gamma = \bigcap_{n=0}^{\infty} \mathcal{F}^n(\bar{\mathcal{U}}) = \bigcap_{n=0}^{\infty} \mathcal{F}^{np}(\bar{\mathcal{U}})$. Thus, on each fiber, $\mathcal{U}$ gives a trapping region for $\mathcal{F}^p$ for the attractor which is the intersection of Γ with that fiber. Thus $\Lambda_i = \bigcap_{n=0}^{\infty} (f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i)^n(\bar{U}_i)$ and Λ_i is an attractor. □

Theorem 4. *If $\mathcal{F}$ is a homeomorphism on a trapping region for an attractor Γ then all the corresponding Λ_i are homeomorphic and the compositions $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i$ on the Λ_i are topologically conjugate.*

Proof. Assume $\mathcal{F}$ is a homeomorphism on trapping region $\mathcal{U}$. Then f_i is a homeomorphism from $\pi_X(\mathcal{U} \cap X_i)$ into $\pi_X(\mathcal{U} \cap X_{i+1 \bmod p})$ for each $i = 0, 1, \ldots, p-1$. Since the composition of homeomorphisms is a homeomorphism, $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i$ is a homeomorphism from $\pi_X(\mathcal{U} \cap X_i)$ into itself. The attractor Γ produced by the trapping region $\mathcal{U}$ is invariant under $\mathcal{F}$ and a subset of $\mathcal{U}$. Thus $\mathcal{F}$ is a homeomorphism from Γ onto Γ. Since $\mathcal{F}$ sends fibers to fibers, the fibers of Γ are homeomorphic. Now $\Lambda_i = \pi_X(\Gamma \cap X_i)$ and π_X restricted to X_i is a homeomorphism. Thus the Λ_i are homeomorphic using $f_i(\Lambda_i) = \Lambda_{i+1}$.

To see that $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i : \Lambda_i \to \Lambda_i$ is topologically conjugate to $f_{i+p} \circ \cdots \circ f_{i+1} \circ f_{i+1} : \Lambda_{i+1} \to \Lambda_{i+1}$ remember that $f_{i+p} = f_i$, so $(f_{i+p} \circ \cdots \circ f_{i+1}) \circ f_i = f_i \circ (f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i)$. Since f_i is a homeomorphism from Λ_i onto Λ_{i+1}, this equation shows that f_i is a topological conjugacy between $f_{i+p} \circ \cdots \circ f_{i+1} \circ f_{i+1}$ and $f_{i+p-1} \circ \cdots \circ$

$f_{i+1} \circ f_i$. Similarly all the $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i$, $i = 0, 1, \ldots, p-1$, are topologically conjugate. □

We will now present several examples of attractors in time-period dynamical systems.

Example 5. This is a 2-periodic system on $\mathbb{R}$. Let $f_0(x) = x/2$ and $f_1(x) = 1 + x/2$. Since $f_0(4/3) = 2/3$ and $f_1(2/3) = 4/3$, the set $\{2/3, 4/3\}$ is invariant. Using the fact that both of these maps are contractions, we see that $2/3$ is the attracting fixed point for $f_0 \circ f_1$ and $4/3$ is the attracting fixed point for $f_1 \circ f_0$. There are many choices for trapping regions. One choice is to take $U_0 = U_1 = (-10, 10)$.

This example illustrates the next theorem about contractions on complete metric spaces.

Theorem 6. *Let $\{f_0, f_1, f_2, \ldots, f_{p-1}\}$ be a p-periodic dynamical system on a complete, locally compact, metric space X. If each f_i is a contraction then each $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i$ is a contraction with a unique fixed point q_i and, for each of these compositions, there is an open set which is a trapping region. The collection of fixed points $\{q_0, q_1, \ldots, q_{p-1}\}$ is an attractor for the p-periodic system.*

Proof. For each i, $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i$ is a contraction since the composition of two contractions is a contraction. Also since a contraction has a unique fixed point, for each i, $f_{i+p-1} \circ \cdots \circ f_{i+1} \circ f_i$ has a unique fixed point q_i. The union of the open unit disks about each q_i is a trapping region with the collection of fixed points $\{q_0, q_1, \ldots, q_{p-1}\}$ as the attractor for the p-periodic system. □

When we are dealing with contractions, we are in a setting similar to that of iterative function systems. The major difference is that in iterative function systems the order of applying the functions is random, while a p-periodic dynamical system has a fixed order for applying the maps.

Example 7. Consider the 2-periodic system on $\mathbb{R}$ where $f_0(x) = x^2 + 1$ and $f_1(x) = x^2 - 1$. Note that these maps are not contractions. Now $f_0(0) = 1$ and $f_1(1) = 0$ so $\{0, 1\}$ is an invariant set for this system. To see that this set is an attractor, we will look at the derivatives to show that we have local attraction. $f_0'(0) = 0$ and $f_1'(1) = 2$, so $(f_0 \circ f_1)'(1) = 0$ and $(f_1 \circ f_0)'(0) = 0$. Since these functions are C^1, we can find trapping regions around 0 and 1. In fact, we can show that $U = (-0.25, 0.25) \cup (0.9, 1.1)$ will work. If we start at 1 with f_0 then the orbit is $\{1, 2, 3, 10, 99, \ldots\}$, which is easily seen to go unbounded. This example shows that we must keep track of the starting time in the trapping regions.

Example 8. Numerically the 2-periodic prey–predator model (2.1) with $\alpha = 0.1$ or $\alpha = 0.4$ has an attractor consisting of two closed curves. Each of these closed curves is an attractor for an autonomous system consisting of the composition of the two maps making up this model. When $\alpha = 0.1$ the trapping regions for these attracting loops can be taken as the same set, i.e., an annular band containing both loops. This is not the case when $\alpha = 0.4$. When $\alpha = 0.4$ the two loops have separated significantly and the union of two narrow

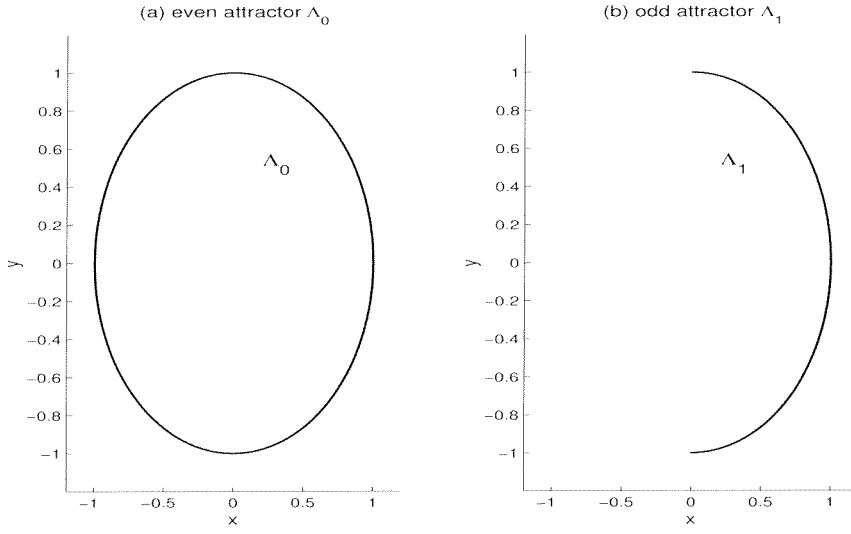

Fig. 5. Nonhomeomorphic subsets Λ_0 and Λ_1.

annuli each containing one loop, is a trapping region. If we start in the diamond-shaped region in Fig. 2 at $t = 0$, the orbit can oscillate wildly and escape to negative infinity.

Example 9. We studied numerically a 5-periodic system in Section 2. In this system the attractor consisted of five closed curves (see Fig. 3). Each of the five closed curves is an attractor for a specific composition of the 5 maps making up this system. There is a large annular region which contains all 5 closed curves and is a trapping region for each of the 5 compositions. This annular region is a trapping region for the 5-periodic system.

If one f_i is not a homeomorphism on a neighborhood of Λ_i then the subsets Λ_i may not be homeomorphic.

Example 10. Take $f_0(x, y) = (|x|, y)$ and $f_1(r, \theta) = (0.5r + 0.5, 2\theta)$. Nontrivial circles centered at the origin are sent onto nontrivial circles centered at the origin by $f_1 \circ f_0$. Under iteration the radius of these circles converge to 1. The attractor for $f_1 \circ f_0$ is Λ_0 the circle of radius 1 and any annulus centered at the origin and containing the unit circle is a trapping region. Because f_0 folds along the y-axis, the attractor for $f_0 \circ f_1$ is the semicircle Λ_1. See Fig. 5.

4. Perturbations of autonomous systems

The p-periodic predator–prey model in Section 2 can be viewed as a perturbation of an autonomous system (i.e., Eq. (2.1) with $\alpha = 0$) with constant predator growth rate. Periodic variation in an intrinsic parameter and periodic forcing are natural scenarios for obtaining p-periodic systems from autonomous systems. The attractor for the autonomous

system (2.1) with $\alpha = 0$ is an invariant closed curve. For the values of α we study, the attractor for the time-dependent model is a union of 2 closed curves (see Fig. 1). In the next two sections we will investigate whether or not an attractor for a perturbed, time-periodic system inherits properties from the attractor of the original autonomous system.

We define the concept of two p-periodic systems being C^0 close. First let $f, g : X \to X$ and let $Y \subset X$. We say the f and g are C^0 ε-close on Y if $d(f(x), g(x)) < \varepsilon$ for all $x \in Y$. Let $\{f_0, f_1, \ldots, f_{p-1}\}$ and $\{g_0, g_1, \ldots, g_{p-1}\}$ be p-periodic dynamical systems of X. Let $Y = \{Y_0, Y_1, \ldots, Y_{p-1}\}$ be a sequence of p subsets of X. Then the p-periodic systems are C^0 ε-close on Y if $d(f_i(x), g_i(x)) < \varepsilon$ for all $x \in Y_i$, $i = 0, 1, \ldots, p-1$. We also say that the two systems are C^0 ε-perturbations of each other. We can extend this definition to C^k ε-perturbations. Start by taking $X = \mathbb{R}^n$ and Y an open subset of $\mathbb{R}^n$. Let $D^j f(x)$ be the jth derivative of f at x and $d(D^j f(x), D^j g(x))$ be the maximum of the jth partial derivatives of $f - g$ at x. Now let $d_k(f(x), g(x)) = \max\{d(f(x), g(x)), d(Df(x), Dg(x)), \ldots, d(D^k f(x), D^k g(x))\}$. By replacing d with d_k in the definition of p-periodic systems being C^0 ε-close on Y, we obtain the definition of p-periodic systems being C^k ε-close on Y.

An example of a property which is inherited under C^0 perturbation is the existence of a trapping region. Given an autonomous map $f : X \to X$, consider the p-periodic system using f for each map. Let $\mathcal{F}$ be the corresponding cylinder map. If $U \subset X$ is a trapping region for f then the corresponding trapping region $\mathcal{U}$ for $\mathcal{F}$ is U on each of the fibers, i.e.,

$$\mathcal{U} = \{0, 1, \ldots, p-1\} \times U.$$

Since $\mathcal{U}$ is a trapping region for $\mathcal{F}$, $\varepsilon = 0.5 \min\{d(\mathcal{F}(z), \mathcal{X} \setminus \mathcal{U}) \colon z \in \bar{\mathcal{U}}\}$ is positive. If $\mathcal{G}$ is any C^0 ε-perturbation of $\mathcal{F}$ then $\mathcal{G}(\bar{\mathcal{U}}) \subset \mathcal{U}$ and $\mathcal{U}$ is a trapping region for $\mathcal{G}$. In particular, if the p-periodic system $\{g_0, g_1, \ldots, g_{p-1}\}$ is a C^0 ε-perturbation of $\{f, f, \ldots, f\}$ and $\mathcal{G}$ is the corresponding cylinder map then $\mathcal{G}$ is a C^0 ε-perturbations of $\mathcal{F}$. Thus U is a trapping region for the p-periodic system $\{g_0, g_1, \ldots, g_{p-1}\}$. The Structure theorem asserts that the attractor $\Lambda \subset U$ for the system $\{g_0, g_1, \ldots, g_{p-1}\}$ is the union of p subsets Λ_i. These Λ_i may be different from each other and different from Λ_f, the attractor for the original autonomous map f, as the following example shows.

Example 11. Take $f(r, \theta) = (0.5r + 0.5, 2\theta)$. Then Λ_f is the circle of radius 1, see Fig. 6(a). In Cartesian coordinates, $f(x, y) = (u(x, y), v(x, y))$ is given by

$$u(x, y) = \frac{(1 + (x^2 + y^2)^{0.5})(x^2 - y^2)}{2(x^2 + y^2)},$$
$$v(x, y) = \frac{(1 + (x^2 + y^2)^{0.5})(xy)}{x^2 + y^2}. \tag{4.1}$$

Define the C^1 ε-perturbation $\{g_0, g_1\}$ of the system $\{f, f\}$ by taking $g_0(x, y) = f(x, y)$ and $g_1(x, y) = f(x, y) + (\varepsilon, \varepsilon)$. For $\varepsilon = 0.05$, the subset Λ_1 appears to be an annulus and hence not homeomorphic to the circle (Fig. 6(a) and (c)). Λ_0 has two distinct rings, see Fig. 6(b). Thus Λ_0 and Λ_1 are not homeomorphic to each other. These regions are not homeomorphic to the circle Λ_f nor to one another because f is not a homeomorphism.

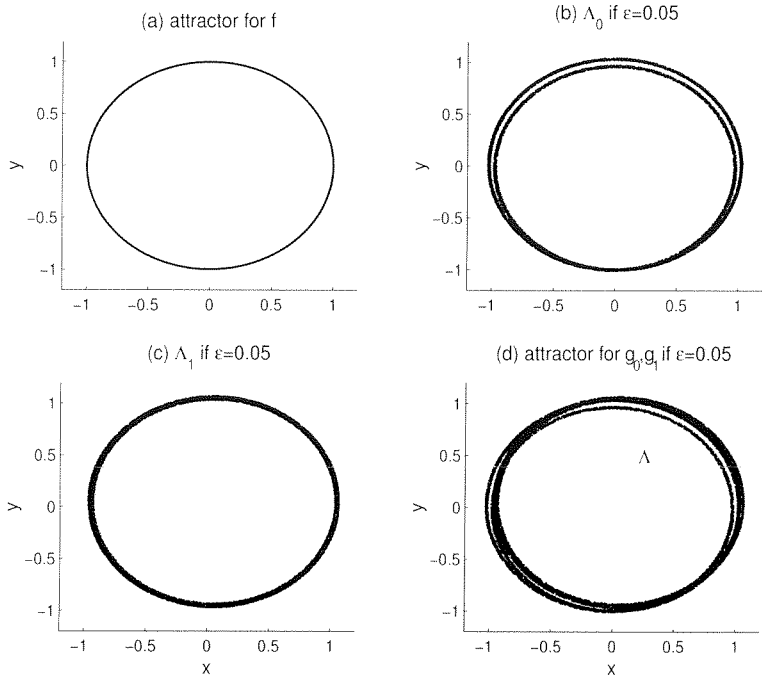

Fig. 6. Subsets Λ_0 and Λ_1 not homeomorphic to the attractor of the autonomous map f.

If f is a diffeomorphism with attractor Λ_f and $\{g_0, g_1, \ldots, g_{p-1}\}$ is a C^1 ε-perturbation of $\{f, f, \ldots, f\}$ then the corresponding attractor Λ for $\{g_0, g_1, \ldots, g_{p-1}\}$ will consist of subsets Λ_i which are homeomorphic to each other. To see this we need the following three results which show that C^1 ε-perturbations preserve diffeomorphic structure. We start with a lemma which is similar to part of the Inverse Function theorem and the proof is motivated by Marsden's proof of the Inverse Function theorem [9]. Let $D(x, r)$ denote the closed disk of radius r centered at x.

Lemma 12. *Let B be an open neighborhood of the origin in $\mathbb{R}^n$. If $f \in C^1(B, \mathbb{R}^n)$, $f(0) = 0$ and $Df(0) = I$ there is a neighborhood N of f in $C^1(B, \mathbb{R}^n)$ and an $r > 0$ such that if $g \in N$ then g is one to one on $D(0, r)$.*

Proof. Let $H_{g,y}(x) = y + x - g(x)$, where $x, y \in B$ and $g \in C^1(B, \mathbb{R}^n)$. Now

$$DH_{f,0}(x)|_{x=0} = D\big(x - f(x)\big)\big|_{x=0} = 0.$$

$DH_{g,0}(x)$ is a continuous function and the components are linear in the partial derivatives of g. So there is an $r > 0$ and a neighborhood N of f in $C^1(B, \mathbb{R}^n)$ such that $x \in D(0, r)$ implies $\|DH_{g,0,i}(x)\| < 1/4n$, for $i = 1, 2, \ldots, n$, where $H_{g,0} = (H_{g,0,1}, H_{g,0,2}, \ldots, H_{g,0,n})$. Taking a slightly smaller N, we may also assume $\|g(0)\| < r/4$.

The Mean Value theorem implies that if $x \in D(0,r)$, then there are points $c_1, c_2, \ldots, c_n \in D(0,r)$ such that

$$H_{g,0,i}(x) - H_{g,0,i}(0) = DH_{g,0,i}(c_i)(x-0) = DH_{g,0,i}(c_i)x.$$

Thus

$$\begin{aligned}\|H_{g,0}(x) - H_{g,0}(0)\| &\leqslant \sum_{i=1}^{n} |H_{g,0,i}(x) - H_{g,0,i}(0)| = \sum_{i=1}^{n} |DH_{g,0,i}(c_i)(x)| \\ &\leqslant \sum_{i=1}^{n} \|DH_{g,0,i}(c_i)\| \|x\| < \frac{\|x\|}{4} < \frac{r}{4}.\end{aligned}$$

The Triangle inequality gives $\|H_{g,0}(x)\| \leqslant \|H_{g,0}(x) - H_{g,0}(0)\| + \|H_{g,0}(0)\| < r/4 + r/4 = r/2$. So $H_{g,0}$ maps $D(0,r)$ into $D(0,r/2)$.

Now let $y \in D(0,r/2)$ and $x \in D(0,r)$. $\|H_{g,y}(x)\| = \|y + H_{g,0}(x)\| \leqslant \|y\| + \|H_{g,0}(x)\| < r/2 + r/2 = r$. So $H_{g,y}: D(0,r) \to D(0,r)$. Let $x_1, x_2 \in D(0,r)$. Then $\|H_{g,y}(x_1) - H_{g,y}(x_2)\| = \|H_{g,0}(x_1) - H_{g,0}(x_2)\| \leqslant \|x_1 - x_2\|/4$ by the Mean Value theorem. So $H_{g,y}$ is a contraction with contraction constant $k = 1/4$. This implies that $H_{g,y}$ has a unique fixed point in $D(0,r)$. Thus

$$p = H_{g,y}(p) = y + p - g(p) \quad \Leftrightarrow \quad g(p) = y$$

Thus there is only one point in $D(0,r)$ that g sends to y.

We finish the lemma by showing g is one to one on $D(0,r/5)$. Let $x \in D(0,r/5)$. Note that $H_{g,0}(x) - H_{g,0}(0) = x - g(x) + g(0)$. So $g(x) - g(0) = x - H_{g,0}(x) + H_{g,0}(0)$. Thus $\|g(x) - g(0)\| \leqslant \|x\| + \|H_{g,0}(x) - H_{g,0}(0)\| \leqslant r/5 + 1/4(r/5) = r/4$ and so $\|g(x)\| \leqslant \|g(0)\| + r/4 \leqslant r/2$. So by the previous paragraph is no other point in $D(0,r)$ that g sends to $g(x)$. Thus g is one to one on $D(0,r/5)$. □

Theorem 13. *Let $A \subset \mathbb{R}^n$ be compact and B be an open neighborhood of A. If $f: B \to \mathbb{R}^n$ is a diffeomorphism onto its image, then there is an open set U with $A \subset U \subset B$ and a neighborhood N of f in the C^1 maps from B into $\mathbb{R}^n$, $C^1(B, \mathbb{R}^n)$, such that if $g \in N$ then g restricted to U is a diffeomorphism onto its image.*

Proof. We start by noting that we can assume that U has compact closure $\bar{U}$ that is contained in B. Since f is a diffeomorphism on B, $|\det(Df(x))| > 0$ on B and bounded away from 0 on $\bar{U}$. The continuity of the partials of the functions in $C^1(B, \mathbb{R}^n)$ and of $|\det(Df(x))|$ on these partials gives a neighborhood N_1 of f in $C^1(B, \mathbb{R}^n)$ such that if $g \in N_1$ then $\det(Dg(x)) \neq 0$ for $x \in \bar{U}$. The Inverse Function theorem tells us that these g are locally one to one and that locally their inverses are C^1 and have nonsingular derivatives.

The remaining step is to show that if the g is close enough to f, then g will be globally one to one on $\bar{U}$. Fix an $x \in \bar{U}$. After a change of coordinates, Lemma 12 can be used to show that there is a neighborhood of f and a compact disk containing x in its interior such that all the g in the neighborhood of f are one to one on this fixed compact disk. Since we can do this for each point in $\bar{U}$, we have a family of compact disks. Now consider a new family D of compact disks obtained from the original family by reducing the radius

of each disk to half its original value. Since $\bar{U}$ is compact, we can cover $\bar{U}$ with a finite number of the compact disks in D. Let r be the minimum radius. Let g be in the finite intersection of the corresponding neighborhoods of f in $C^1(B, \mathbb{R}^n)$. If $x, y \in \bar{U}$ and they are less than r apart, then they are in one of our original disks and $g(x) \neq g(y)$.

Now $(f, f): \bar{U} \times \bar{U} \to \mathbb{R}^n \times \mathbb{R}^n$ with the diagonal going to the diagonal and these are the only points going to the diagonal. Now if we restrict (f, f) to $\bar{U} \times \bar{U}$ minus the r-neighborhood of the diagonal, the image is compact and misses the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ by some distance $\varepsilon > 0$. If g is within $\varepsilon/2$ of f then (g, g) on $\bar{U} \times \bar{U}$ minus the r-neighborhood of the diagonal also misses the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$. So if $x, y \in \bar{U}$ and they are at least r apart, then $g(x) \neq g(y)$. We now have a neighborhood N of f in $C^1(B, \mathbb{R}^n)$, such that if $g \in N$ then g is one to one on $\bar{U}$. Thus these g are diffeomorphisms from U onto their images. □

Theorem 14. *Let $\{f_0, f_1, \ldots, f_{p-1}\}$ be a p-periodic dynamical system with corresponding cylinder map $\mathcal{F}$. Let $\mathcal{U}$ be a trapping region for $\mathcal{F}$. If $\mathcal{F}$ is a diffeomorphism on a neighborhood of $\bar{\mathcal{U}}$, then there is an $\varepsilon > 0$ and a neighborhood $\mathcal{V}$ of $\bar{\mathcal{U}}$ such that any p-periodic system C^1 ε-close to $\mathcal{F}$ is a diffeomorphism on $\mathcal{V}$.*

Proof. $\mathcal{F}$ is a diffeomorphism on some open neighborhood $\mathcal{B}$ of $\bar{\mathcal{U}}$. The intersection of $\mathcal{B}$ and $\bar{\mathcal{U}}$ with each fiber in the cylinder space $\mathcal{X}$ gives an open neighborhood of a compact set in $\mathbb{R}^n$. Now $\mathcal{F}$ restricted to this open set is precisely the situation of the last theorem. Thus each of the functions $f_0, f_1, \ldots, f_{p-1}$ are diffeomorphisms from these corresponding open sets onto their images. If ε is less than the distance between the fibers in the fibered cylinder space, then any C^0 perturbation of $\mathcal{F}$ of size less than ε will send the fibers to the fibers in the same order. Thus the perturbation can be thought of as coming from a perturbation of the p-periodic dynamical system. In fact close to $\mathcal{F}$, $C^1(\mathcal{X}, \mathcal{X})$ has the product structure of $C^1(X, X) \times C^1(X, X) \times \cdots \times C^1(X, X)$. Applying Theorem 13 to each f_i, we see that on each fiber there is an open set $\mathcal{V}_i$ in $\mathcal{B}$ which contains the intersection of $\bar{\mathcal{U}}$ with the corresponding fiber and a neighborhood N_i of f_i such that if $g_i \in N_i$ then g_i is a diffeomorphism on $\mathcal{V}_i$. The cylinder map $\mathcal{G}$ that $\{g_0, g_1, \ldots, g_{p-1}\}$ generates is a diffeomorphism of

$$\bigcup_{i=0}^{p-1} \{i\} \times \mathcal{V}_i = \mathcal{V}. \qquad \square$$

Example 15. Take $f(x) = \arctan(x)$. Then 0 is a globally attracting fixed point for the diffeomorphism f. To get a 2-periodic dynamical system that is C^1 close to $\{f, f\}$ let $g_0(x) = \arctan(x)$ and $g_1(x) = (1 + a)\arctan(x)$, where $a > 0$. The attractor for $\{f, f\}$ is the single fixed point 0. By the Structure theorem the attractor for $\{g_0, g_1\}$ consists of the union of the attractors Λ_0 for $g_1 \circ g_0$ and Λ_1 for $g_0 \circ g_1$. These attractors are symmetric intervals centered at 0. Their union is the larger of the two intervals. When $a = 0.1$, $g_1 \circ g_0$ has three fixed points, $\{-0.3997, 0, 0.3997\}$ and $g_0 \circ g_1$ also has three fixed points, $\{-0.3802, 0, 0.3802\}$. Hence, the attractors are $\Lambda_0 = [-0.3997, 0.3997]$ and $\Lambda_1 = [-0.3802, 0.3802]$. We see that the two attractors are homeomorphic to each other but they are not homeomorphic to the original attractor for f.

From the last example we see that a small C^1 perturbation of a diffeomorphism can produce an attractor for a time-periodic dynamical system with attracting subsets which are not homeomorphic to the attractor for the diffeomorphism. In the next section we will develop hyperbolic structure to obtain sufficient conditions so that C^1 ε-perturbations will have homeomorphic attractors.

5. Hyperbolic cycles

If X is an open subset of $\mathbb{R}^n$ or an n-dimensional manifold then a fixed point $x_0 \in X$ of $f : X \to X$ is *hyperbolic* if the derivative $Df(x_0)$ has no eigenvalues on the unit circle. A p-cycle $\{x_0, x_1, \ldots, x_{p-1}\}$ of f is *hyperbolic* if x_0 is a hyperbolic fixed point of the map f^p. In fact, if $\{x_0, x_1, \ldots, x_{p-1}\}$ is a hyperbolic p-cycle then for each point x_i, $i = 0, 1, \ldots, p-1$, the matrix $Df^p(x_i)$ has no eigenvalue on the unit circle because the eigenvalues of $Df^p(x_i)$ are the same as those of $Df^p(x_0)$. A hyperbolic p-cycle of f produces a hyperbolic p-cycle for the corresponding cylinder map as described in the next paragraph.

Let $\{g_0, g_1, \ldots, g_{p-1}\}$ be a p-periodic dynamical system on X and $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ be the corresponding cylinder map. Since $\mathcal{X}$ is p copies of X topologically, $\mathcal{X}$ is a n-dimensional manifold. The derivative of $\mathcal{G}$ at $(i, x) \in \mathcal{X}$ is just $D\mathcal{G}(i, x) = Dg_i(x)$. A p-cycle $\{(0, x_0), (1, x_1), \ldots, (p-1, x_{p-1})\}$ of $\mathcal{G}$ is hyperbolic if $(0, x_0)$ is a hyperbolic fixed point of the map $\mathcal{G}^p$. Clearly, $D\mathcal{G}^p(0, x_0)$ is given by the product

$$Dg_{p-1}(x_{p-1}) \cdots Dg_1(x_1) Dg_0(x_0).$$

Hence, if $\{x_0, x_1, \ldots, x_{m-1}\}$ is a hyperbolic m-cycle of f and m divides p then $\{(0, x_0), (1, x_1), \ldots, (p-1, x_{p-1})\}$ is a hyperbolic p-cycle of the cylinder map $\mathcal{F}$ associated to the constant p-periodic system $\{f, f, \ldots, f\}$ because

$$D\mathcal{F}^p(0, x_0) = Df(x_{p-1}) \cdots Df(x_1) Df(x_0) = Df^p(x_0).$$

Recall that hyperbolic p-cycles are stable under C^1 perturbation. For a discussion of this result for maps which are not diffeomorphisms, see Appendices 1 and 4 in Palis and Takens [10]. If $\{g_0, g_1, \ldots, g_{p-1}\}$ is a C^1 ε-perturbation of $\{f, f, \ldots, f\}$ with cylinder map $\mathcal{G}$ then $\mathcal{G}$ is a C^1 ε-perturbation of $\mathcal{F}$. For $i = 0, \ldots, m-1$, each point x_i on the hyperbolic m-cycle of f determines a hyperbolic p-cycle of $\mathcal{F}$ starting with $(0, x_i)$, i.e., $\{(0, x_i), (1, x_{i+1}), \ldots, (p-1, x_{i+p-1})\}$. Hence, there is a y_i near x_i and a hyperbolic p-cycle of $\mathcal{G}$ starting at $(0, y_i)$. Thus, the hyperbolic m-cycle of f produces m hyperbolic p-cycles of the C^1 ε-perturbation $\{g_0, g_1, \ldots, g_{p-1}\}$. This result is analogous to a result of Henson [5] where the perturbation corresponds to small amplitude p-periodic forcing. Henson [5] exhibits an example where the perturbed p-cycles are not distinct.

The notion of hyperbolicity may be extended to compact invariant sets of a map (see, e.g., Palis and Takens [10]). Let $f : X \to X$ be a C^1 map and $\Lambda \subset X$ be a compact invariant set for f. Λ is *hyperbolic* if there are constants $A > 0$ and α, $0 < \alpha < 1$, and a continuous splitting of the tangent bundle over Λ into stable and unstable subbundles, i.e., $T_\Lambda X = E^s \oplus E^u$, so that

(i) $Df(E^s) \subset E^s$ and $\|(Df_{|E^s})^n\| \leqslant A\alpha^n$ for all $n \geqslant 1$,
(ii) $Df(E^u) = E^u$, $Df_{|E^u}$ has an inverse and $\|(Df_{|E^u})^{-n}\| \leqslant A\alpha^n$ for all $n \geqslant 1$.

If Λ_f is a hyperbolic invariant set for a diffeomorphism f and g is a C^1 ε-perturbation of f then g has a hyperbolic invariant set Λ_g homeomorphic to Λ_f and $f_{|\Lambda_f}$ is topologically conjugate to $g_{|\Lambda_g}$, (see Theorem 7.4 in [12]). However, if f is not invertible in a neighborhood of Λ_f then g still has a hyperbolic invariant set Λ_g close to Λ_f [10] but $f_{|\Lambda_f}$ may not be conjugate to $g_{|\Lambda_g}$ and Λ_f and Λ_g may not be homeomorphic. For an example in the context of p-periodic dynamical systems, see Example 11 and Fig. 6.

Theorem 16. *Let f have a hyperbolic attractor Λ_f and be a diffeomorphism on a neighborhood of $\bar{U}$ the closure of a trapping region producing Λ_f. Then there is an $\varepsilon > 0$ and an open neighborhood V of $\bar{U}$ such that if $\{g_0, g_1, \ldots, g_{p-1}\}$ is a C^1 ε-perturbation of $\{f, f, \ldots, f\}$ on V then V is a trapping region for $\{g_0, g_1, \ldots, g_{p-1}\}$ and produces an attractor consisting of p subsets each of which is homeomorphic to Λ_f.*

Proof. If f is a diffeomorphism in a neighborhood of Λ_f then the cylinder map $\mathcal{F}$ corresponding to the p-periodic system $\{f, f, \ldots, f\}$ is a diffeomorphism in a neighborhood of the hyperbolic invariant set $\Gamma_{\mathcal{F}}$, where $\Gamma_{\mathcal{F}}$ is Λ_f in each fiber. If $\{g_0, g_1, \ldots, g_{p-1}\}$ is a C^1 ε-perturbation of $\{f, f, \ldots, f\}$ with cylinder map $\mathcal{G}$ then $\mathcal{G}$ is a C^1 ε-perturbation of $\mathcal{F}$ and also $\mathcal{G}$ is a diffeomorphism because of Theorem 14. It follows that $\mathcal{G}$ has a hyperbolic invariant set $\Gamma_{\mathcal{G}}$ homeomorphic to $\Gamma_{\mathcal{F}}$. Moreover, because the homeomorphism between $\Gamma_{\mathcal{G}}$ and $\Gamma_{\mathcal{F}}$ is fiberwise, the corresponding invariant set $\Lambda_g = \pi_X(\Gamma_{\mathcal{G}})$ for the p-periodic system $\{g_0, g_1, \ldots, g_{p-1}\}$ consists of subsets each homeomorphic to Λ_f. □

6. Summary

In order to study attractors for a p-periodic, discrete dynamical system $\{f_0, f_1, \ldots, f_{p-1}\}$ on a state space X, this work introduces the cylinder space $\mathcal{X}$ by adding time as a state variable and introduces the corresponding cylinder map $\mathcal{F}$. Because $\mathcal{F}$ is autonomous and $\mathcal{X}$ is compact in the time direction, an attractor Λ for $\{f_0, f_1, \ldots, f_{p-1}\}$ may be defined as the projection of an attractor for $\mathcal{F}$. The Structure theorem shows that Λ is the union of p subsets Λ_i, where each Λ_i is an attractor of an autonomous system formed by composing all of the f_i taken in an appropriate order. These Λ_i are homeomorphic if each f_i is a homeomorphism. Example 10 (see Fig. 5) presents an attractor where the Λ_i are not homeomorphic because one f_i is not a homeomorphism.

Periodic variation in an autonomous system f with attractor Λ_f often results in a p-periodic system with attractor Λ. However, Example 11 (see Fig. 6) shows that if f is not one to one then even a small C^1 perturbation may have an attractor where the subsets Λ_i are not homeomorphic to each other nor to Λ_f. As a result of Theorems 4 and 14, if f is a diffeomorphism in a neighborhood of Λ_f then the subsets Λ_i are homeomorphic to each other but may not be homeomorphic to Λ_f (see Example 15). With the additional assumption of hyperbolicity on Λ_f, we show that the Λ_i are also homeomorphic to Λ_f.

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