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Compatibility, Development, and Estimation of Taper and Volume Equation Systems

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The meaning of compatibility in systems of taper and volume equations has been extended. It is desirable and possible to develop completely compatible taper and volume equation systems that have algebraic compatibility and numeric consistency among all the component equations. Two such taper and volume systems were developed for slash pine in the southeastern United States. One was derived from a recently developed merchantable volume equation (Zhao and Kane 2017), and the other was derived from the well-known Max and Burkhart segmented polynomial taper equation (1976). Three fitting methods were used to obtain numerically consistent estimates of parameters in these two systems. The performance of the systems associated with fitting methods were evaluated with taper, merchantable volume and total volume predictions, and bias trends over different dbh classes, total height classes, and different relative height intervals. The new system outperformed the Max-Burkhart taper-based system for merchantable and total volume predictions and was competitive in diameter prediction. Optimization of parameters of the system for both taper and cumulative volume simultaneously is preferable to separate optimization for taper or volume only.

Keywords: variable-top merchantable volume, taper equation, nonlinear seemingly unrelated regressions, generalized nonlinear least squares

The development of tree volume and taper equations has a long history. Volume tables and equations were produced more than 200 years ago (c.f., Clark 1902), and the form and taper of tree stems have been studied for more than 100 years (c.f., Beher 1923). The theory of compatible taper and volume equation systems was first introduced in the early 1970s (Demaerschalk 1971, 1972). Since then, the meaning of compatibility in taper and volume equation systems has been expanded, and several approaches for developing compatible systems of taper and volume equation have been proposed.

As initially defined by Demaerschalk (1971, 1972, 1973), compatible taper equations, when integrated, produce an identical estimate of total volume to that given by existing volume equations. When Clutter (1980) developed taper functions from variable-top merchantable volume equations, compatibility meant that integrating taper equations along the length of any section of the tree bole must produce the volume of that section obtained from the corresponding variable-top merchantable volume equations. Now, the concept of compatibility in a system of taper and volume equations has been expanded so that all component equations (taper, total volume, and merchantable volume) in that system are algebraically compatible with each other. Generally speaking, algebraic compatibility means that parameters of one equation can be written in terms of parameters in another equation. For example, algebraically rearranging parameters of a taper equation gives parameters for a corresponding merchantable volume equation and total volume equation. Conversely, parameters of a taper equation and a total volume equation can be obtained from the parameters of a merchantable volume equation. In such an algebraically compatible system, therefore, the taper equation and merchantable volume equation share the same set of parameters. Furthermore, estimation of parameters in an algebraically compatible system should ensure numeric consistency among the component equations. In other words, all component equations in that system also share the same parameter estimates. Thus, we can define a completely compatible taper and volume equation system as one that has both the algebraic compatibility and the same parameter estimates.

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Two common approaches are available to develop algebraically compatible taper and volume equations. One is to develop taper equations first, then integrate the taper equation to get merchantable volume and total volume equations. More complicated taper equations may be more accurate for predicting stem taper, such as the Kozak (2004) variable-exponent taper equation (Li and Weiskittel 2010), but deriving closed-form compatible merchantable and total volume equations from some complex taper equations is almost impossible. The second approach is to derive compatible taper equations from volume equations by solving a differential equation after differentiating that volume equation (e.g., Clutter 1980, Bailey 1994) or by differentiating the volume equation only (e.g., Van Deusen et al. 1982, Lynch et al. 2017), depending on the volume model forms. No matter which form a merchantable volume model (V_m) has, it can always be expressed as total stem volume (V_T) multiplied by the ratio of merchantable to total volume (R) (see Appendix A). Thus, deriving compatible taper equations from merchantable volume equations essentially becomes deriving compatible taper equations from the volume ratio functions (R).

Deriving compatible taper equations from upper diameter (d)-based volume ratio equations (R_d) requires the solution of a differential equation after differentiating that volume ratio R_d with respect to upper height (h). Thus, deriving closed-form compatible taper equations is not difficult from R_ds with power functions of d (e.g., Clutter 1980, Bailey 1994, Fang and Bailey 1999, Fang et al. 2000), but almost impossible from R_ds with exponential functions of d [except in Jordan et al. (2005) with a complicated Incomplete Gamma function]. However, taking the partial derivative of upper height-based volume ratios (R_h) with respect to h directly leads to compatible taper equations (Van Deusen et al. 1982, Reed and Green 1984, Lynch et al. 2017). Whether R_h is a power function or an exponential function of h, it is possible to derive closed-form compatible taper equations. For given merchantable volume equations, there is a corresponding uniquely defined compatible taper function (Clutter 1980). The resultant taper and volume equation cannot guarantee algebraic compatibility if the general volume ratio function R (R_d or R_h) does not satisfy four conditions (see Appendix A).

To ensure numeric consistency among component equations in an algebraically compatible system, three methods can be used to obtain parameter estimates. The first method is to fit the taper equation to taper measurement data and then extract the parameters for the merchantable volume and the total volume equations from the fitted taper equation (Goulding and Murray 1976). The second method is to fit the merchantable volume equation to cumulative volume data first, then extract the parameters for the total volume and taper equations from the fitted merchantable volume equation (Lynch et al. 2017, Zhao and Kane 2017). The third method is to simultaneously fit the taper equation and merchantable volume equation to the taper measurement data and cumulative volume data of the same sampled trees (Van Deusen 1988, Fang et al. 2000). To the best of our knowledge, there is no study exploring how these methods change the system performance in predicting taper, merchantable volume, and total volume. Some researchers separately fit a volume equation and a taper equation, then adjust the coefficients of the taper equation so that the summation of volumes of tree segments derived from the taper equation is equal to the tree volume as estimated from the volume equation (e.g., Munro and Demaerschalk 1974), or the resulting taper model produces predictions that match dbh and/or total volume predicted separately from the total volume equation (e.g., Özçelik and Cao 2017). These conditioning methods would not necessarily improve the fit of the taper equation (Özçelik and Cao 2017).

This study compared two algebraically compatible systems of taper and volume equations: one was derived from a newly published merchantable volume equation (Zhao and Kane 2017) and the other was derived from the well-known Max and Burkhart (1976) segmented polynomial taper equation with the quadratic-quadratic-quadratic (Q-Q-Q) form. The systems, along with three different parameter estimation methods, were evaluated in terms of taper, merchantable volume, and total volume predictions. The objective was to develop a completely compatible system that can more accurately and precisely predict total volume, merchantable volume, and diameter at any specified height for slash pine (*Pinus elliottii* Engelm.), a leading timber species in the southeastern United States.

Data

Data used in this study are from two data sets of measurements on slash pine trees sampled from plantation plots in the coastal plain of Georgia and north Florida. The first data were obtained from 134 trees felled in 2013–2016 and the second data were from 1,289 trees felled in 1993, 2012, and 2013. Four sample trees without any obvious abnormalities were felled on each sample plot. Two trees classified as dominant or codominant were selected from a large dbh class, a third tree was selected from the average dbh class, and a fourth tree was selected from a dbh class smaller than the average dbh class. Each sample tree was measured for dbh with a diameter tape and, after felling, for total height with a measuring tape. The sample trees in the first data set were cut leaving a 0.15-m stump, then diameter outside bark (cm) at 0.15, 0.61, 1.22, 2.44, 3.66, 4.88 m . . . up to a top diameter of <5.1 cm was measured. The felled trees in the second data set were cut into 1.52-m bolts starting at ground level up to a top diameter of <5.1 cm. Each bolt was measured for diameter outside bark at the small end. Total tree volume outside bark and cumulative volume from the butt to successive bolt heights were calculated using the overlapping bolts method of Bailey (1995). Distribution of all sampled trees by dbh and total height classes is shown in Table 1.

Methods

Systems of Taper and Volume Equations

The following notation will be used throughout the rest of this article. Other notations specific to a particular equation will be listed with the equation.

D: dbh (cm);

H: total tree height (m);

h: height above ground to diameter d (m);

d: upper stem diameter, outside bark (cm);

Management and Policy Implications

The developed completely compatible taper and volume equation systems with parameters optimized simultaneously for both taper and cumulative volume can provide accurate and consistent estimations of taper, merchantable volume, and total volume for slash pines in the southern United States.

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Table 1. Distribution of all sampled trees by DBH and total height classes.

DBH class (cm)	Total height class (m)											Total
	6	8	10	12	14	16	18	20	22	24	26	
6	2	3										5
8	3	46	49	9								107
10		23	42	44	9							118
12		13	34	55	41	13	2					158
14		3	19	44	60	34	18	1				179
16			12	35	41	38	29	7				162
18			2	17	53	52	61	23	11	1		220
20				1	20	69	44	34	10			178
22					8	21	31	32	12	8	1	113
24				1	1	6	11	20	9	7		55
26						2	16	40	14	4		76
28						2	2	20	12	4		40
30								5	2	1	1	9
32									1			1
34								1	1			2
Total	5	88	158	206	233	237	214	183	72	25	2	1423

p : stem relative height, $p = h / H$;

V_T : total tree outside-bark volume (m^3);

V_m : volume from the ground to some top diameter or height limit (i.e., merchantable volume) (m^3);

V_h : merchantable volume up to height h (m^3);

R : volume ratio, V_m / V_T ; ratio which when multiplied by total tree volume gives merchantable volume;

R_p : volume ratio for V_m prediction to a relative height (p);

R_h : volume ratio for V_m prediction to an upper height limit (h);

R_d : volume ratio for V_m prediction to an upper diameter limit (d);

k : a constant, $\pi / 40\,000$ for metric units or $\pi / 576$ for English measurement units.

The first system was derived from the following volume ratio function, which is the best model form selected from eight new upper height-based volume ratio models proposed by Zhao and Kane (2017):

$$R_p = \left[1-(1-p)^{\alpha}\right]^{\beta};\ \alpha > 1,\ 0 < \beta \le 1. \tag{1}$$

This volume ratio function satisfies all four conditions (I–IV). In general, a compatible taper equation can be derived from the volume ratio R_p or R_h (Lynch et al. 2017):

$$d(h) = \sqrt{(V_T / k)(\partial R_p / \partial h)}. \tag{2}$$

The parameter β in Eq. 1 is strongly related to tree volume, and Eq. 1 could be modified as

$$R_p = \left[1-(1-p)^{\alpha}\right]^{\left\{1-\theta \exp\left[-\exp\left(\phi D^{a_1} H^{a_2}\right)\right]\right\}},\ \alpha > 1. \tag{3}$$

Thus, the corresponding merchantable volume equation could be written as

$$V_h = a_0 D^{a_1} H^{a_2}\left[1-\left(1-\frac{h}{H}\right)^{\alpha}\right]^{\left\{1-\theta \exp\left[-\exp\left(\phi D^{a_1} H^{a_2}\right)\right]\right\}}. \tag{4}$$

Here, the tree total volume equation is

$$V_T = a_0 D^{a_1} H^{a_2}. \tag{5}$$

The compatible taper function derived from Eq. 4 is

$$d(h) = \sqrt{\begin{array}{l}\dfrac{\alpha}{k} a_0 D^{a_1} H^{a_2 - 1}\left(1-\dfrac{h}{H}\right)^{\alpha-1} \\ \left\{1-\theta \exp\left[-\exp\left(\phi D^{a_1} H^{a_2}\right)\right]\right\} \\ \left[1-\left(1-\dfrac{h}{H}\right)^{\alpha}\right]^{\left\{-\theta \exp\left[-\exp\left(\phi D^{a_1} H^{a_2}\right)\right]\right\}}\end{array}}. \tag{6}$$

Eq. 5 was originally presented by Schumacher and Hall (1933) in the logarithmic form. Eqs. 4–6 constitute the first system of taper and volume equations, which is algebraically compatible among component equations. That is, Eq. 4 and Eq. 6 share the same set of parameters a_0, a_1, a_2, α, ϕ, and θ, and share a_0, a_1, and a_2 with Eq. 5.

The second system was derived from the well-known Max and Burkhart (1976) six-parameter Q-Q-Q taper equation. The Max-Burkhart taper equation was rewritten in the following form:

$$d(h) = D\sqrt{\begin{array}{l} b_1\left(\dfrac{h}{H}-1\right)+b_2\left(\left(\dfrac{h}{H}\right)^2-1\right) \\ +b_3\left(a_1-\dfrac{h}{H}\right)^2 I_1 + b_4\left(a_2-\dfrac{h}{H}\right)^2 I_2\end{array}} \tag{7}$$

where:

$$I_1 = \begin{cases} 1 & \text{if } \frac{h}{H} \le a_1 \\ 0 & \text{if } \frac{h}{H} > a_1 \end{cases} \qquad I_2 = \begin{cases} 1 & \text{if } \frac{h}{H} \le a_2 \\ 0 & \text{if } \frac{h}{H} > a_2 \end{cases}.$$

Integrating Eq. 7 results in the following merchantable volume equation:

$$\begin{aligned} V_h = kD^2 H &\left\{\frac{b_3 a_1^3 + b_4 a_2^3}{3} + \frac{b_1}{2}\left(\frac{h}{H}\right)^2 + \frac{b_2}{3}\left(\frac{h}{H}\right)^3\right. \\ &\left. -(b_1+b_2)\left(\frac{h}{H}\right) - \frac{b_3}{3}\left(a_1-\frac{h}{H}\right)^3 I_1 - \frac{b_4}{3}\left(a_2-\frac{h}{H}\right)^3 I_2\right\}. \end{aligned} \tag{8}$$

A total volume equation was derived by setting $h = H$:

$$V_T = k\left(\frac{b_3 a_1^3 + b_4 a_2^3}{3} - \frac{b_1}{2} - \frac{2b_2}{3}\right) D^2 H. \tag{9}$$

Eq. 9 is a special case of the constant-form factor volume equation (Spurr 1952). Eqs. 7–9 comprise an algebraically compatible taper and volume equation system.

Fitting Methods

The systems of taper and volume equations derived above consist of three interdependent components: the taper equation, the merchantable volume equation, and the total volume equation. Due to their algebraic compatibility, three fitting methods were used to obtain parameter estimates and to ensure numeric consistency among the component equations.

Fitting Method 1 (FM-1)—Fitting to Taper Data

The taper equation was fit using weighted nonlinear least squares (WNLS). First, the taper equation in each system was fit to the

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taper measurement data using nonlinear least squares. Second, the estimated errors from the first step were modeled as a power function of D, H, and h, yielding a weight function for heteroscedasticity (Parresol 2001, Zhao et al. 2015). Third, after fixing the weight function and refitting the taper equation using the WNLS, an autoregressive function AR(2) was determined to be best to describe the correlation between successive measurements on the same tree. Last, the taper equation was refit using the WNLS with the fixed weight function and an AR(2) correlation structure. After finally refitting the taper equation to taper measurement data, parameter estimates for the merchantable volume equation and the total volume equation in the system were extracted from the fitted taper equation, respectively.

FM-2—Fitting to Cumulative Volume Data

The merchantable volume equation was first fitted to cumulative volume data using the nonlinear least squares. In the same manner used for fitting the taper equations above, a weight function was developed to address heteroscedasticity, and a better autoregressive function [i.e., AR(2)] was found to account for the correlation between successive measurements on the same tree. Finally, the merchantable volume equation was refit with WNLS using the fixed weight function and the AR(2) correlations. After finally refitting the merchantable volume equation to cumulative volume data, parameter estimates for the corresponding taper equation and the total volume equation were calculated or extracted from the fitted merchantable volume equation.

FM-3—Simultaneously Fitting to Taper and Cumulative Volume Data

First, the merchantable volume equation and taper equation in each system were simultaneously fitted to taper measurement data and cumulative volume data of the same sample trees using nonlinear seemingly unrelated regression (NSUR), which accounts for the inherent correlation between the taper and volume equations. Then, based on the estimated errors of the unweighted NSUR, a unique weight function as a power function of D, H, and h was formed for each equation, and an autoregressive function AR(2) was found appropriate for both taper and merchantable volume equations. Finally, after fixing the weight functions, the merchantable volume and taper equation system were simultaneously refit using weighted NSUR with the AR(2) structure to estimate all parameters.

When $h = H$ in the original data, observed $d(h) = 0$ and observed merchantable volume are equal to observed total tree volume. It is theoretically correct that when h is equal to H in the two systems, the merchantable volume equations (Eqs. 4 and 8) will convert to the corresponding total volume equations (Eqs. 5 and 9), respectively. In the first system, the merchantable volume equation (Eq. 4) contains $1 - h/H$. The nonlinear parameter estimation algorithm makes use of logarithms of $1 - h/H$, and the logarithm will be undefined when $h = H$. Thus, the observations of total volume will be ignored and cannot be used directly in FM-2 and FM-3. It is not reasonable to estimate parameters to predict total volume without having the total volume observations involved in the parameter estimation processes (Fang et al. 2000). To solve this problem in the parameter estimation processes, the merchantable volume equation in the first system was rewritten as:

$$V_h = (1 - \gamma)V_h + \gamma V_T, \tag{10}$$

where $\gamma = 1$, if $h = H$; otherwise $\gamma = 0$. The merchantable volume equation V_h and the total volume equation V_T are defined by Eqs. 4 and 5 in the first system, respectively. Similarly, the taper equations were rewritten as:

$$d(h) = (1 - \gamma)d(h), \tag{11}$$

where the taper equations are defined by Eqs. 6 or 7.

All the model fits were performed using the MODEL Procedure in the SAS/ETS 9.3 (SAS Institute, Inc. 2011).

Model Assessment and Evaluation

The common statistics of average error (E), relative error (RE), standard error of estimate (SEE), relative standard error of estimate ($RSEE$), and the coefficient of determination (R^2) were calculated for upper stem diameter, cumulative volume, and total tree volume to evaluate the taper and volume equation systems with the three fitting methods.

To more closely examine the effectiveness of the systems associated with the fitting methods, the average errors of estimated stem diameter, cumulative volume, and total tree volume were determined by dbh class, total stem height class, and relative height interval from the ground and were compared graphically.

Results and Discussion

As shown in the Appendix A, a uniquely defined compatible taper equation could be derived from a given volume ratio function, regardless of R_d, or R_h. The methodologies used to derive such a compatible taper equation from R_d (e.g., Clutter 1980, Bailey 1994) or from R_h (e.g., Van Deusen et al. 1982, Reed and Green, 1984, Lynch et al. 2017) can be unified by taking the partial derivative of merchantable volume equations with respect to h. To achieve the algebraic compatibility in a system of taper and volume equations, the volume ratio function in that system (or implied in that system) should satisfy all four conditions (see the Appendix A). However, most previously published volume ratio functions do not satisfy all four conditions. For example, volume ratio models used by Burkhart (1977), Cao and Burkhart (1980), Pienaar et al. (1985, 1987), Gregoire and Schabenberger (1996), and Bullock and Burkhart (2003) do not satisfy the first condition. Zhao and Kane (2017) proposed several upper height-based volume ratio functions that satisfy all four conditions, and their corresponding compatible taper equations are given in Lynch et al. (2017).

Parameters for the two algebraically compatible systems were estimated by FM-1, FM-2, and FM-3, respectively, to ensure numeric consistency among component equations. There are some differences in parameter estimates and some big differences in parameter standard errors due to the fitting methods (Tables 2 and 3). Parameters in the system were optimized only for taper in FM-1 and were optimized for cumulative volume and total volume in FM-2. In FM-3 all parameters in the system were optimized for taper, cumulative volume, and total volume. When fitting the merchantable equation in the first system using FM-2 and FM-3, the technique described above (Eq. 10) was used to avoid undefined logarithms so that the total volume observations were used in the parameter estimation processes.

The residual variances were stabilized with the weight functions (Tables 2 and 3). The correlations between successive measurements on the same tree were described as an AR(2) structure. The measurements

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Table 2. Parameter estimates and their standard errors (italic values) and the associated weight function for the first system of slash pine taper and volume equations from three approaches.

Fitting method	Weight function	$\hat{a}_0$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}$	$\hat{\theta}$	$\hat{\phi}$
FM-1 (Eq. 6)	$D^{2.1079}H^{-0.3863}h^{-0.1698}$	5.7763E-5	1.9608	0.8955	2.0850	0.4170	4.2003E-6[b]
		1.5567E-6	*0.0112*	*0.0130*	*0.0059*	*0.0050*	*1.5940E-6*
FM-2 (Eq. 4)	$D^{3.0014}H^{0.2872}h^{1.2658}$	4.7636E-5	1.8633	1.0749	2.0601	0.2617	-8.0094E-6[a]
		6.1402E-7	*0.0044*	*0.0055*	*0.0113*	*0.0187*	*2.6803E-6*
FM-3 (Eqs. 4 & 6)	$D^{2.1003}H^{-0.4144}h^{-0.1966}$	4.9917E-5	1.8728	1.0496	2.0706	0.4057	8.0034E-6
	$D^{3.1862}h^{1.3369}$	*5.2597E-7*	*0.0038*	*0.0048*	*0.0045*	*0.0044*	*1.1208E-6*

In fitting method FM-1, the taper equation (Eq. 6) is fitted to taper data, then the estimated parameters are substituted into the merchantable volume equation (Eq. 4) and the total volume equation (Eq. 5). In fitting method FM-2, Eq. 4 is fitted to cumulative volume data, the estimated parameters are substituted into Eq. 6 and Eq. 5. In fitting method FM-3, the six parameters are estimated by simultaneously fitting Eqs. 4 and 6 to cumulative volume and taper data.
[a] $p = 0.0019$; [b] $p = 0.0084$; others $p < 0.0001$.

Table 3. Parameter estimates and their standard errors (italic values) and the associated weight function for the second system of slash pine taper and volume equations from three approaches.

Fitting method	Weight function	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$	$\hat{b}_4$	$\hat{a}_1$	$\hat{a}_2$
FM-1 (Eq. 7)	$D^{1.6109}H^{-0.5883}h^{-0.0765}$	-4.2532	1.9819	-2.0342	1.1471E2	0.7857	7.8018E-2
		0.0878	*0.0481*	*0.0476*	*3.7231*	*0.0043*	*1.1223E-3*
FM-2 (Eq. 8)	$D^{3.2165}H^{-0.8457}h^{1.2288}$	-5.7504	2.8048	-2.9796	1.4466E2	0.8113	7.1691E-2
		0.9616	*0.5232*	*0.5098*	*28.2517*	*0.0221*	*4.5179E-3*
FM-3 (Eqs. 7 & 8)	$D^{1.5476}H^{-0.4919}h^{0.0626}$	-4.3273	2.0237	-2.0908	1.2308E2	0.7867	7.5676
	$D^{3.0795}H^{-0.8554}h^{1.2504}$	*0.0869*	*0.0475*	*0.0467*	*3.7813*	*0.0041*	*1.0085E-3*

In fitting method FM-1, the taper equation (Eq. 7) is fitted to taper data, then the estimated parameters are substituted into the merchantable volume equation (Eq. 8) and the total volume equation (Eq. 9). In fitting method FM-2, Eq. 8 is fitted to cumulative volume data, the estimated parameters are substituted into Eq. 7 and Eq. 9. In fitting method FM-3, the six parameters are estimated by simultaneously fitting Eqs. 7 and 8 to taper measurement data and cumulative volume data.
All $p < 0.0001$.

Table 4. Overall performance comparison of taper and volume equation systems associated with the parameter estimation methods, based on taper, merchantable volume and total volume predictions.

System	Fitting method	Taper					Merchantable volume					Total volume				
		E (cm)	RE (%)	SEE (cm)	RSEE (%)	R^2	E (m^3)	RE (%)	SEE (m^3)	RSEE (%)	R^2	E (m^3)	RE (%)	SEE (m^3)	RSEE (%)	R^2
1	FM-1	0.142	1.23	0.990	8.62	0.980	0.005	3.05	0.019	11.03	0.984	0.007	3.37	0.023	11.01	0.980
	FM-2	-0.102	-0.89	0.986	8.58	0.980	0.003	1.96	0.017	9.71	0.988	0.002	1.07	0.020	9.72	0.985
	FM-3	-0.046	-0.40	0.958	8.34	0.981	0.001	0.55	0.017	9.64	0.988	0.002	0.80	0.020	9.70	0.985
2	FM-1	0.059	0.51	0.835	7.27	0.985	-0.009	-4.97	0.022	12.62	0.979	-0.006	-2.83	0.024	11.68	0.978
	FM-2	0.072	0.62	0.848	7.38	0.985	0.004	2.58	0.017	10.09	0.987	0.005	2.23	0.022	10.34	0.983
	FM-3	0.074	0.65	0.835	7.27	0.985	0.004	2.21	0.017	10.03	0.987	0.005	2.17	0.022	10.32	0.983

Note: $E = \Sigma(y_i - \hat{y}_i)/n$, $RE = E/\bar{y}\times 100$, $SEE = \sqrt{\Sigma(y_i - \hat{y}_i)^2/(n-p)}$, $RSEE = SEE/\bar{y}\times 100$, $R^2 = 1 - \Sigma(y_i - \hat{y}_i)^2/\Sigma(y_i - \bar{y})^2$. Since the six parameters in each of the two systems are shared by the taper, merchantable volume, and total volume equations, p is 3 in this case.

of the two data sets in this study were not conducted with the same spacing between measurements. Some correlation structures with continuous functions of distance between measurements might perform better. But the preliminary analysis suggested that their improvement was minimal compared with AR(2) for this specific study.

Overall, both systems performed well with $R^2 > 0.978$ for all taper, merchantable volume, and total volume predictions (Table 4). Given a fitting method, the first system provided better prediction for merchantable and total volumes, and the second system provided a little better prediction for taper, in terms of *E*, *RE*, *SEE*, *RSEE*, and R^2. Both FM-2 and FM-3 improved cumulative and total volume predictions and slightly improved taper prediction compared to FM-1 for the first system. For the second system, both FM-2 and FM-3 improved cumulative and total volume predictions but did not change the performance of the taper equation. Given a system, the estimation methods could be ranked based on either the errors or standard errors of estimates of taper, merchantable volume, and total volume (Kozak and Smith 1993). For the first system, either by overall average errors or by overall standard errors of estimates, FM-3 ranked first and FM-1 ranked

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third (Table B.1 in Appendix B). For the second system, FM-3 also ranked first and FM-1 ranked third by overall standard errors of estimates; the fitting methods were equally ranked by overall errors.

The local performances of a given system were changed by the parameter estimation methods (Figures 1 and 2). For the first system, it was a little surprising that even when the taper equation

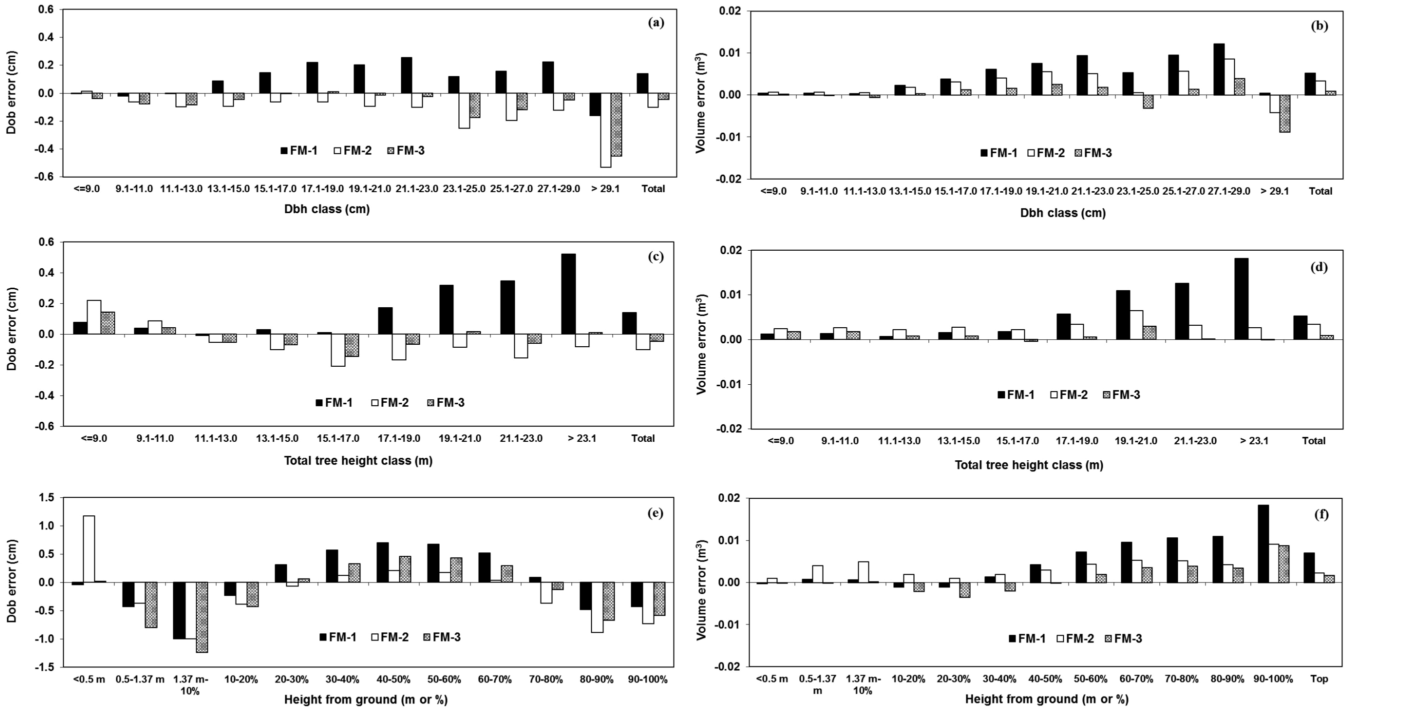

Figure 1. Comparisons of the average errors of estimated outside-bark diameter and merchantable volume by dbh class, total height class, relative height from ground using the first system fitted by FM-1 (fitting to taper data), FM-2 (fitting to cumulative volume data), and FM-3 (simultaneously fitting to taper and cumulative volume data).

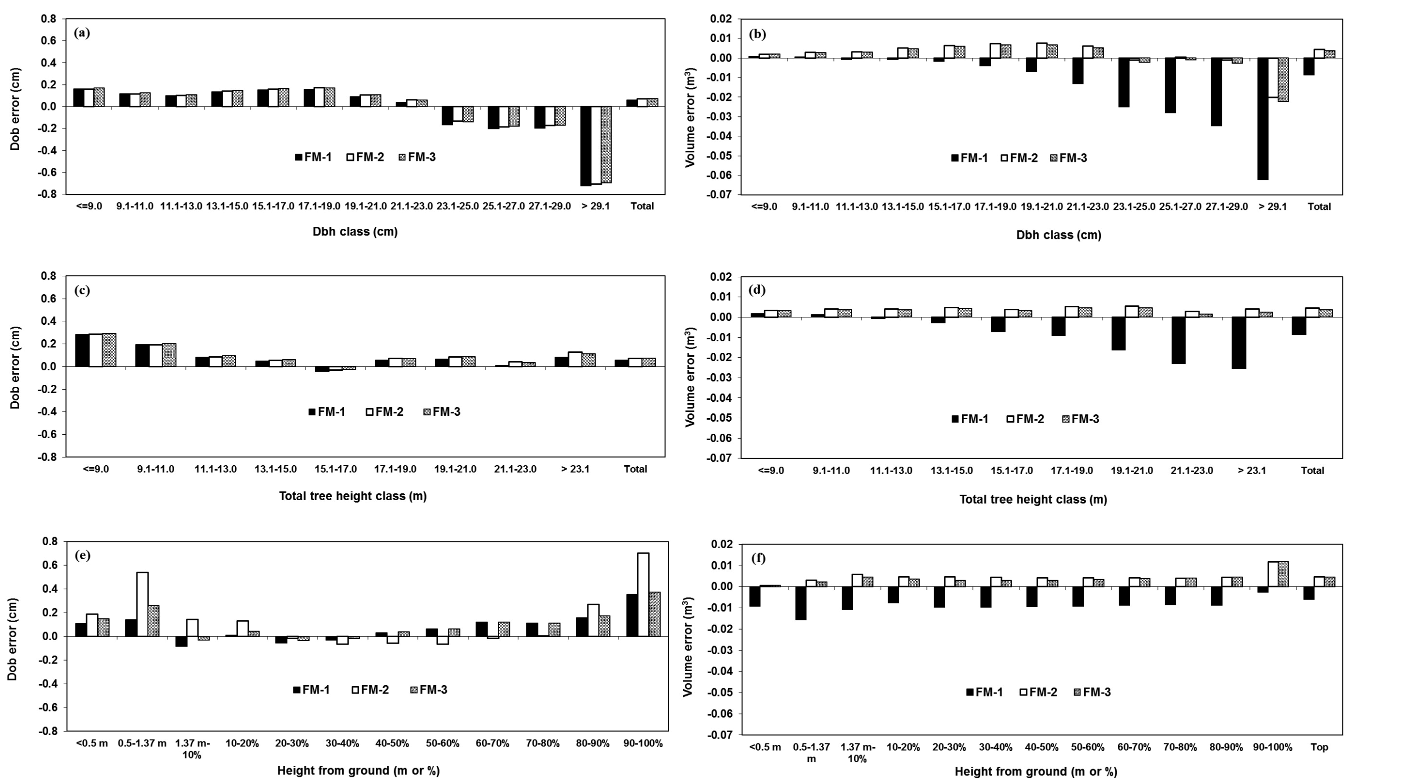

Figure 2. Comparisons of the average errors of estimated outside-bark diameter and merchantable volume by dbh class, total height class, relative height from ground using the second system fitted by FM-1 (fitting to taper data), FM-2 (fitting to cumulative volume data), and FM-3 (simultaneously fitting to taper and cumulative volume data).

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was fitted to taper data, the fitted taper equation generally did not provide the best prediction for taper (Figure 1a, c, e), although it might have small diameter errors for small trees (dbh <13 cm or total height <17 m), trees in some large DBH classes >23 cm, or near the bottom and at top sections (<20% and >80% of total height). Furthermore, the first system with FM-1 provided the worst prediction for merchantable volume, having larger errors for large trees or on the upper half of the stem (Figure 1b, d, f), and total volume (Table 4). This suggests that it might not be a good practice to estimate cumulative volume or total volume based only on the developed taper equation with FM-1. For the first system, it is clear that FM-3 performed better than FM-1 and FM-2 in terms of merchantable volume, total volume, and taper predictions.

For the second system, FM-1, FM-2, and FM-3 produced very similar results for taper prediction over dbh classes and total tree height classes (Figure 2a, c). FM-1 and FM-3 also had very similar diameter error patterns at stem portion from ground to top, but FM-2 had relatively higher diameter errors near the butt (<1.37 m) and at top sections of tree (>80% total height) (Figure 2e). FM-1 produced the worst results for merchantable volume prediction (Figure 2b, d, f) and total volume prediction (Table 4) among the three estimation methods. There were no large differences between FM-2 and FM-3 in merchantable and total volume predictions.

Compared with FM-1, FM-2 and FM-3 substantially improved the estimation of merchantable volumes for both systems, because the magnitudes in average errors at various portions were smaller than for FM-1. With FM-2 or FM-3, the second system systematically underestimated merchantable volume from the ground to the top. For the second system, the change from systematic overestimation of merchantable volume with FM-1 to underestimation of merchantable volume with FM-2 or FM-3 is partly because the total volume equation in that system is a constant form factor function (Eq. 9). This total volume equation with parameters calculated from the fitted taper equation (Eq. 7) overestimated total volume and had a smaller value of R^2 (0.978) and larger values of *SEE* (0.024) and *RSEE* (11.68%) (Table 4). The Schumacher and Hall (1933) volume equation in the first system (Eq. 5) provided better estimates of the total volume ($R^2 = 0.980$, *SEE* = 0.023 and *RSEE* = 11.01%) even with parameters extracted from the fitted taper equation (Eq. 6).

The taper and merchantable volume equations in an algebraically compatible system share the same parameters, so the simultaneous estimation approach is recommended (Fang et al. 2000) if taper, merchantable (i.e., cumulative) volume, and total volume observation are available. Our results further supported this recommendation. In the theory of seemingly unrelated regressions, the seemingly unrelated regression estimation is generally more precise for estimating the parameters than separate estimation of equations in the system if the equations have correlated errors. Our results showed that FM-3 had a lower standard error or an equivalent standard error, though there were a few exceptions. Of course, sometimes the difference was not very much (Tables 2 and 3).

Using FM-3, the two favored systems fit reasonably well. More than 98.1% of the variation about the mean values of *d*, V_m, and V_T is explained by the model systems (Table 4). Overall average error and *SEE* of total volume are 0.002 m³ (0.80%) and 0.020 m³ (9.70%) for the first system, and 0.005 m³ (2.17%) and 0.022 m³ (10.32%) for the second system (Table 4; or relative height 100% in Table 7). The two systems associated with FM-3 were ranked on the errors and standard errors of estimates for outside bark diameter, merchantable and total volumes for different dbh classes (Table 5), for different total height classes (Table 6), and at different relative height levels from ground (Table 7). Our new system (i.e., the first one) ranked first (Table B.2 in Appendix B).

Table 5. Performance comparison of model systems associated with FM-3 and by dbh classes for outside bark diameter and merchantable volume for slash pine.

System	Dbh class (cm)	n	Upper diameter				Merchantable volume			
			E (cm)	RE (%)	SEE (cm)	RSEE (%)	E (m³)	RE (%)	SEE (m³)	RSEE (%)
1	<9.0	716	-0.037	-0.68	0.503	9.17	0.0003	1.54	0.0019	11.56
1	9.1–11.0	890	-0.076	-1.14	0.595	8.94	-0.0000	-0.05	0.0038	11.72
1	11.1–13.0	1345	-0.083	-1.07	0.700	9.03	-0.0006	-1.14	0.0062	11.67
1	13.1–15.0	1757	-0.047	-0.53	0.749	8.49	0.0003	0.39	0.0091	11.11
1	15.1–17.0	1667	0.000	0.00	0.827	8.31	0.0012	1.09	0.0112	10.06
1	17.1–19.0	2624	0.012	0.10	0.866	7.61	0.0016	1.05	0.0134	8.68
1	19.1–21.0	2284	-0.015	-0.12	0.990	7.82	0.0026	1.32	0.0179	9.18
1	21.1–23.0	1578	-0.024	-0.17	1.114	7.86	0.0018	0.72	0.0211	8.36
1	23.1–25.0	830	-0.173	-1.12	1.124	7.31	-0.0032	-1.05	0.0242	7.96
1	25.1–27.0	1092	-0.120	-0.72	1.402	8.40	0.0014	0.39	0.0283	7.81
1	27.1–29.0	604	-0.049	-0.27	1.430	7.93	0.0040	0.93	0.0276	6.47
1	>29.1	180	-0.449	-2.32	1.611	8.33	-0.0088	-1.73	0.0311	6.07
2	<9.0	716	0.170	3.10	0.564	10.28	0.0020	11.72	0.0028	16.45
2	9.1–11.0	890	0.125	1.88	0.604	9.07	0.0027	8.20	0.0046	14.19
2	11.1–13.0	1345	0.109	1.41	0.676	8.73	0.0030	5.60	0.0065	12.38
2	13.1–15.0	1757	0.147	1.67	0.726	8.23	0.0048	5.87	0.0103	12.58
2	15.1–17.0	1667	0.164	1.65	0.782	7.86	0.0060	5.36	0.0126	11.30
2	17.1–19.0	2624	0.171	1.50	0.768	6.75	0.0068	4.38	0.0148	9.58
2	19.1–21.0	2284	0.106	0.84	0.876	6.93	0.0068	3.50	0.0191	9.78
2	21.1–23.0	1578	0.059	0.42	0.921	6.50	0.0052	2.08	0.0211	8.39
2	23.1–25.0	830	-0.141	-0.92	0.922	5.99	-0.0021	-0.70	0.0236	7.79
2	25.1–27.0	1092	-0.178	-1.07	1.113	6.67	-0.0010	-0.27	0.0278	7.66
2	27.1–29.0	604	-0.171	-0.95	1.100	6.10	-0.0026	-0.60	0.0268	6.28
2	>29.1	180	-0.694	-3.59	1.373	7.10	-0.0222	-4.34	0.0408	7.98

Percent *E* (*RE*) and percent *SEE* (*RSEE*) are based on the average observed values of the dbh class.

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Table 6. Performance comparison of model systems associated with FM-3 and by height classes for outside bark diameter and merchantable volume for the slash pine data.

System	Total height class (m)	n	Upper diameter				Merchantable volume			
			E (cm)	RE (%)	SEE (cm)	RSEE (%)	E (m^3)	RE (%)	SEE (m^3)	RSEE (%)
1	<9.0	544	0.145	2.29	0.595	9.40	0.0018	8.63	0.0031	14.55
1	9.1–11.0	1145	0.044	0.60	0.689	9.39	0.0018	4.80	0.0047	12.67
1	11.1–13.0	1744	-0.052	-0.61	0.753	8.86	0.0008	1.32	0.0066	10.60
1	13.1–15.0	2293	-0.069	-0.69	0.835	8.38	0.0008	0.78	0.0101	10.14
1	15.1–17.0	2693	-0.146	-1.30	0.937	8.31	-0.0004	-0.30	0.0132	8.97
1	17.1–19.0	2710	-0.064	-0.54	0.993	8.30	0.0006	0.31	0.0175	8.99
1	19.1–21.0	2629	0.016	0.11	1.157	7.99	0.0030	1.05	0.0235	8.16
1	21.1–23.0	1247	-0.060	-0.39	1.177	7.64	0.0000	0.01	0.0260	7.89
1	>23.1	562	0.010	0.06	1.125	6.83	-0.0001	-0.03	0.0262	7.13
	<9.0	544	0.294	4.65	0.669	10.56	0.0032	15.26	0.0042	19.98
2	9.1–11.0	1145	0.202	2.75	0.692	9.43	0.0039	10.62	0.0059	15.97
2	11.1–13.0	1744	0.095	1.12	0.706	8.29	0.0038	6.07	0.0075	12.01
2	13.1–15.0	2293	0.063	0.63	0.755	7.58	0.0043	4.32	0.0108	10.80
2	15.1–17.0	2693	-0.025	-0.22	0.822	7.29	0.0033	2.22	0.0138	9.42
2	17.1–19.0	2710	0.072	0.60	0.861	7.20	0.0046	2.39	0.0182	9.35
2	19.1–21.0	2629	0.085	0.59	0.985	6.80	0.0045	1.58	0.0253	8.78
2	21.1–23.0	1247	0.034	0.22	0.949	6.16	0.0016	0.47	0.0267	8.09
2	>23.1	562	0.113	0.69	0.848	5.15	0.0026	0.71	0.0225	6.14

Percent *E* (*RE*) and percent *SEE* (*RSEE*) are based on the average observed values of the total height class.

Table 7. Performance comparison of model systems associated with FM-3 and at several heights from ground level for outside bark diameter and merchantable volume for the slash pine data.

System	Height from ground	n	Upper diameter				Merchantable volume			
			E (cm)	RE (%)	SEE (cm)	RSEE (%)	E (m^3)	RE (%)	SEE (m^3)	RE (%)
1	<0.50 m	1004	0.020	0.10	1.236	5.97	-0.0001	-1.43	0.0009	12.94
1	0.51–1.37 m	263	-0.802	-3.66	1.020	4.66	-0.0002	-0.30	0.0038	7.44
1	1.38 m-10%	595	-1.243	-6.36	1.378	7.05	0.0002	0.36	0.0075	11.41
1	10%-20%	1500	-0.428	-2.69	0.825	5.18	-0.0021	-2.86	0.0076	10.09
1	20%-30%	1440	0.066	0.42	0.822	5.23	-0.0035	-2.82	0.0106	8.56
1	30%-40%	1528	0.329	2.29	0.935	6.48	-0.0021	-1.31	0.0125	7.95
1	40%-50%	1454	0.458	3.51	1.008	7.73	-0.0002	-0.12	0.0143	7.80
1	50%-60%	1428	0.436	3.72	1.030	8.79	0.0019	0.93	0.0174	8.32
1	60%-70%	1495	0.293	2.91	0.962	9.57	0.0035	1.54	0.0201	8.84
1	70%-80%	1438	-0.126	-1.56	0.920	11.44	0.0039	1.62	0.0221	9.07
1	80%-90%	1338	-0.673	-11.84	1.155	20.33	0.0034	1.30	0.0231	8.83
1	90%-100%	661	-0.580	-17.47	1.023	30.80	0.0087	3.01	0.0261	9.01
1	100%	1423	0.000	0.00	0.000	0.00	0.0017	0.80	0.0203	9.70
2	<0.50 m	1004	0.149	0.72	1.294	6.25	0.0005	7.56	0.0011	14.80
2	0.51–1.37 m	263	0.260	1.19	0.730	3.33	0.0023	4.43	0.0043	8.40
2	1.38 m-10%	595	-0.029	-0.15	0.559	2.86	0.0046	6.96	0.0089	13.47
2	10%-20%	1500	0.043	0.27	0.700	4.39	0.0035	4.68	0.0076	10.06
2	20%-30%	1440	-0.032	-0.20	0.811	5.16	0.0030	2.42	0.0104	8.38
2	30%-40%	1528	-0.018	-0.13	0.849	5.89	0.0030	1.88	0.0130	8.26
2	40%-50%	1454	0.038	0.29	0.852	6.54	0.0030	1.62	0.0151	8.25
2	50%-60%	1428	0.063	0.54	0.904	7.71	0.0035	1.66	0.0181	8.65
2	60%-70%	1495	0.120	1.19	0.913	9.08	0.0039	1.70	0.0206	9.08
2	70%-80%	1438	0.113	1.40	0.916	11.39	0.0041	1.70	0.0225	9.22
2	80%-90%	1338	0.173	3.04	0.880	15.50	0.0046	1.76	0.0243	9.27
2	90%-100%	661	0.376	11.33	0.753	22.68	0.0118	4.06	0.0278	9.60
2	100%	1423	0.000	0.00	0.000	0.00	0.0046	2.17	0.0216	10.32

Percent *E* (*RE*) and percent *SEE* (*RSEE*) are based on the average observed values of the relative height class.

Fang et al. (2000) developed an eight-parameter system of compatible inside bark taper and volume equations for slash pine based on segmented-stem form factor equations. Our systems had six parameters fitted to outside bark taper and volume for slash pine. Even though we did not fit Fang's system to our data, we could still compare approaches using the relative values of some fitting statistics. The relative error and *SEE* of total inside bark volume in the system of Fang et al. (2000) are 0.38% and 12.46%, respectively. Across dbh and total height classes and from ground to top, the first system generally had smaller values of the relative error and *SEE* of merchantable volume than either Fang's system or our second system. The relative error and *SEE* of upper diameter across dbh and total height classes and from ground to top indicated that the first system can be competitive with the second system and Fang's system for modeling slash pine taper, although the first system had larger errors at heights >80% of total height. Lynch et al. (2017) also found that Eq. 6 in the first system and Eq. 7 in the second system were competitive in performance when fitted to loblolly pine taper data.

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Although the first one appears to have a more complex nonlinear form, the two systems are about equally complex because both have six parameters. The system of Fang et al. (2000) has eight parameters and is more complex but did not perform better than our first one.

A better taper equation not only needs to predict stem form well but also provide accurate estimates of stem volume (Li and Weiskittel 2010). Here, stem volume should include variable-top merchantable volume and total stem volume. More complicated taper equations might be more accurate for predicting stem taper, but it is difficult to derive closed-form compatible merchantable and total volume equations from them, as discussed in Appendix A. When taper equations do not have corresponding closed-form merchantable and total volume equations, their usefulness is limited in practice due to the difficulty of implementing them for volume estimation. This is one reason why most studies have compared the accuracy of stem taper equations in predicting taper only, rather than in total volume and especially in merchantable volume. For a given species, comparison of taper equation forms oftentimes found that some forms performed better in predicting diameters and some other forms were superior in estimating stem volume. It is reasonable to develop a taper equation that performs better for both taper and volume rather than to select one for diameter, another for volume. To achieve this goal, developing compatible taper and volume equations could be a good choice.

Previous studies either fitted taper equations to taper data only (Lynch et al. 2017), fitted merchantable volume equations to cumulative volume only (Bailey 1994), or simultaneously fitted taper and merchantable volume equations to taper and cumulative volume data (Coble and Hilpp 2006, Özçelik and Cao 2017). A very limited number of studies (e.g., Fang and Bailey 1999, Fang et al. 2000) have simultaneously fitted all component equations in the system. In our current study, we comprehensively compared three parameter estimation methods including simultaneous estimation approach, all of which ensure numeric consistency among component equations in algebraically compatible systems. In addition to goodness-of-fit statistics, the system should be evaluated in predicting upper diameter and merchantable volume across dbh classes, total height classes, and at relative heights from ground to top.

Compatible taper and volume systems offer many benefits such as additivity and flexibility over traditional taper equations (Fang et al. 2000, Li and Weiskittel 2010). However, in the current study, we demonstrate that keeping taper equation and merchantable and total volume equations algebraically compatible does not guarantee accurate estimates of all components in the system (taper, merchantable, and total volumes). The performance of a given algebraically compatible system could be changed by parameter estimation methods, even though these methods ensure numeric consistency among component equations in that system.

Conclusions

The meaning of compatibility of taper and volume equation systems is being extended. A completely compatible system of taper, total, and merchantable volume equations should be algebraically compatible and numerically consistent among all component equations. To achieve the algebraic compatibility of a system either derived from a taper equation or from a merchantable volume equation, there exists a corresponding volume ratio function R that should satisfy all four conditions (see Appendix A). In addition, the taper function should be capable of being integrated over $0 \le h \le H$. At minimum, all the component equations should have a closed form. To ensure numeric consistency in an algebraically compatible taper and volume equation system, parameters in that system could be estimated by FM-1, FM-2, or FM-3, depending on the available data. The performance of taper and volume equation systems need to be evaluated with taper, merchantable volume, and total volume predictions. Our results demonstrated that FM-3 provides the best prediction for both merchantable and total volumes as well as a competitive taper prediction, while FM-1 provides the worst prediction for merchantable and total volumes but does not guarantee the best prediction for taper. Therefore, FM-3 is preferable to the other two methods and FM-1 is not recommended if all data of taper, cumulative volume, and total volume are available. The first system derived from a newly published merchantable volume equation (Zhao and Kane 2017) can be competitive with the second system derived from the well-known Max and Burkhart (1976) Q-Q-Q taper equation for upper-stem diameter prediction but outperformed the Max-Burkhart derived system in merchantable and total volume predictions.

Appendix A

For any type of merchantable volume equation either to an upper diameter limit (d) or to an upper stem height limit (h), theoretically there is a corresponding, uniquely defined compatible taper equation. This compatible taper equation may or may not have a closed form, depending on the function form of d or h associated with the merchantable equation.

Basically, merchantable volume equations could be grouped into three classes. The first one includes $V_m = V_T - V_{top}$ (V_{top}: top volume of tree stem above a diameter d at a distance from the top of the stem) and $V_m = V_T R$, in which V_{top} or R is a function of upper-stem diameter d. This class of merchantable volume equations has been intensively developed and used to derive compatible taper equations. For merchantable volume equations with an exponential function of d or exponential ratio form of R_d (e.g., Van Deusen et al. 1981, Tasissa et al. 1997), the corresponding compatible taper equations either do not have a closed form or are involved with a complicated Incomplete Gamma function (Jordan et al. 2005). For the merchantable volume equation with power functions of d, however, a closed form of compatible taper equation can be derived, as shown below. The second class is referred to as upper stem height-based merchantable volume equations, in which V_{top} or R is a function of upper stem height h, that is, $V_{top}(h)$ or R_h. Differentiating $V_{top}(h)$ or R_h with respect to h can quickly result in a closed form of compatible taper equations (Lynch et al. 2017, Zhao and Kane 2017). In the third class of merchantable volume equations, V_{top} or R are functions of both d and h. A closed-form compatible taper equation could be derived from this class of merchantable volume equation with power functions of both d and h (Bailey 1994).

In the first class of the merchantable volume equations, there are two types of V_m equations associated with power functions of d, from which closed-form compatible taper functions can be derived. One type used by Honer (1964), Burkhart (1977), and Cao and Burkhart (1980) is

$$V_m = V_T[1 - b_1 d^{b_2} D^{b_3}]. \tag{A.1}$$

Compatible taper functions to merchantable volume equations of this type were developed by Clutter (1980). Equation (A.1) could be further extended to a general form as

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$$V_m = V_T[1 - b_1 d^{b_2} D^{b_3} H^{b_4}]. \quad (A.2)$$

Another type of variable top-diameter merchantable volume equations is defined as total volume minus top volume. The general form is

$$V_m = V_T - V_{top} = a_0 D^{a_1} H^{a_2} - a_3 d^{a_4} D^{a_5} H^{a_6}. \quad (A.3)$$

A.3 and its compatible taper equations have been used by Pienaar et al. (1985). A.3 can be rewritten as the same form as A.2

$$V_m = a_0 D^{a_1} H^{a_2} (1 - b_1 d^{b_2} D^{b_3} H^{b_4}) = V_T (1 - b_1 d^{b_2} D^{b_3} H^{b_4})$$
$$(b_1 = a_3 / a_0,\ b_2 = a_4,\ b_3 = a_5 - a_1,\ b_4 = a_6 - a_2). \quad (A.4)$$

Therefore, compatible taper equations for both types of merchantable volume equations (A.2 and A.3) can be derived in the same way.

Let R_d denote the volume ratio for V_m prediction to any upper diameter limit and R_h denote the volume ratio for V_m prediction to any upper height limit. A.2 – A.4 can be written as $V_m = V_T R_d$, where $R_d = 1 - b_1 d^{b_2} D^{b_3} H^{b_4}$. By the definition of the taper function, it should be noted that $d^2 = d^2(h) = f(h)$. Then,

$$R_d = 1 - b_1 d^{b_2} D^{b_3} H^{b_4} = 1 - b_1 [f(h)]^{\frac{b_2}{2}} D^{b_3} H^{b_4} = R_h. \quad (A.5)$$

Thus, A.2 can be rewritten as $V_m = V_T R_d = V_T R_h$. It is because of the possibility of deriving compatible R_h from R_d (see A.5) or deriving compatible R_d from R_h (Reed and Green 1984) that the methodology for deriving compatible taper equations from a volume ratio R_d or R_h can be unified by taking the derivative of merchantable volume with respect to upper height limit *h*.

That is, either from variable top-diameter merchantable volume equations or from variable top-height merchantable volume equations, the same approach could be followed to derive compatible taper equations:

$$V_T R_h = k \int_0^h d^2(h) \partial h, \quad (A.6)$$

or

$$V_T R_d = k \int_0^h f(h) \partial h. \quad (A.7)$$

where $k = \pi / 40\,000$ for metric units or $k = \pi / 576$ for English measurement units. Differentiating both sides of A.6 with respect to *h* yields

$$V_T \frac{\partial R_h}{\partial h} = k \times d^2(h). \quad (A.8)$$

Solving for $d(h)$ yields a compatible taper equation:

$$d(h) = \sqrt{\frac{V_T}{k} \frac{\partial R_h}{\partial h}}. \quad (A.9)$$

Zhao and Kane (2017) proposed 11 height-based volume ratio functions. All these ratio functions are differentiable with respect to *h*, and the corresponding taper functions are given in Lynch et al. (2017).

Differentiating both sides of A.7 with respect to *h* leads to

$$V_T \frac{\partial R_d}{\partial h} = k \times f(h). \quad (A.10)$$

According to A.5, we can get

$$\frac{\partial R_d}{\partial h} = -b_1 D^{b_3} H^{b_4} \left(\frac{b_2}{2}\right) [f(h)]^{\frac{b_2}{2} - 1} \frac{\partial f(h)}{\partial h}. \quad (A.11)$$

Then, substituting for $\partial R_d / \partial h$ from A.11 into A.10 gives

$$-b_1 \left(\frac{b_2}{2}\right) V_T D^{b_3} H^{b_4} [f(h)]^{\frac{b_2}{2} - 1} \frac{\partial f(h)}{\partial h} = k \times f(h).$$

or, following some rearrangement,

$$-k b_1^{-1} (2 b_2) V_T^{-1} D^{-b_3} H^{-b_4} \partial h = [f(h)]^{\frac{b_2}{2} - 2} \partial f(h). \quad (A.12)$$

A.12 is a separable differential equation involving the variables *h* and $f(h)$. Integration of A.12 leads to

$$-k b_1^{-1} (2 / b_2) V_T^{-1} D^{-b_3} H^{-b_4} h = \left(\tfrac{b_2}{2} - 1\right)^{-1} [f(h)]^{\frac{b_2}{2} - 1} + C. \quad (A.13)$$

When $h = H$, $f(H) = 0$ so that $C = -k b_1^{-1} (2 / b_2) V_T^{-1} D^{-b_3} H^{1 - b_4}$. Making use of this result, the fact that $d(h) = \sqrt{f(h)}$, and some algebraic rearrangement gives

$$d(h) = \left[k b_1^{-1} \left(\tfrac{b_2 - 2}{b_2}\right) V_T^{-1} D^{-b_3} H^{-b_4} (H - h) \right]^{\frac{1}{b_2 - 2}}. \quad (A.14)$$

For the special case of $b_4 = 0$ in merchantable volume equation (A.1), the compatible taper equation as shown by Clutter (1980) is

$$d(h) = \left[k b_1^{-1} \left(\tfrac{b_2 - 2}{b_2}\right) V_T^{-1} D^{-b_3} (H - h) \right]^{\frac{1}{b_2 - 2}}. \quad (A.15)$$

For the merchantable volume equation (A.3) in which $b_1 = a_3 / a_0$, $b_2 = a_4$, $b_3 = a_5 - a_1$, $b_4 = a_6 - a_2$, and $V_T = a_0 D^{a_1} H^{a_2}$, the compatible taper equation is

$$d(h) = \left[k \left(\tfrac{a_0}{a_3}\right) \left(\tfrac{a_4 - 2}{a_4}\right) V_T^{-1} D^{a_1 - a_5} H^{a_2 - a_6} (H - h) \right]^{\frac{1}{a_4 - 2}}. \quad (A.16)$$

or,

$$d(h) = \left[k \left(\tfrac{a_4 - 2}{a_3 a_4}\right) D^{-a_5} H^{-a_6} (H - h) \right]^{\frac{1}{a_4 - 2}}. \quad (A.17)$$

Substituting for $f(h)$ from A.14 into A.5 with $f(h) = d^2(h)$, $\alpha = b_2 / (b_2 - 2)$, and making some algebraic rearrangement leads to

$$R_h = 1 - b_1 [d(h)]^{b_2} D^{b_3} H^{b_4}$$
$$= 1 - \left(b_1 D^{b_3} H^{b_4}\right)^{1 - \alpha} \left(\frac{k}{\alpha V_T}\right)^{\alpha} (H - h)^{\alpha}. \quad (A.18)$$

Recall $V_T = a_0 D^{a_1} H^{a_2}$, and $p = h / H$. Rewriting A.18 as

$$R_p = 1 - b_1 k^{\alpha} \left(\alpha a_0 b_1\right)^{-\alpha} D^{b_3 - \alpha(a_1 + b_3)} H^{b_4 + \alpha(1 - a_2 - b_4)} (1 - p)^{\alpha}. \quad (A.19)$$

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Substituting
$b_0' = b_1 k^{\alpha}(\alpha a_0 b_1)^{-\alpha}$, $b_1' = b_3 - \alpha(a_1 + b_3)$, and $b_2' = b_4 + \alpha(1 - a_2 - b_4)$

into A.19 gives

$$R_p = 1 - b_0' D^{b_1'} H^{b_2'} (1-p)^{\alpha}. \tag{A.20}$$

In summary, A.1 and A.3 are based on upper-diameter, that is, R_d. Their compatible upper height-based volume ratio function R_h has the same form as A.18 or R_p as the form as A. 20.

Let $\beta = b_0' D^{b_1'} H^{b_2'}$, then A.20 becomes

$$R_p = 1 - \beta(1-p)^{\alpha}. \tag{A.21}$$

An upper height-based merchantable volume equation corresponding to A.20 is written as

$$V_h = a_0 D^{a_1} H^{a_2} - c_0 D^{c_1} H^{c_2} \left(1 - \frac{h}{H}\right)^{\alpha}, \tag{A.22}$$

where $c_0 = a_0 \cdot b_0'$, $c_1 = a_1 \cdot b_1'$, and $c_2 = a_2 \cdot b_2'$. Total volume included in A. 22 is

$$V_T = a_0 D^{a_1} H^{a_2}. \tag{A.23}$$

Differentiating the volume ratio R_p in A.20 or merchantable volume V_h in A.22 with respect to h yields the following compatible taper equation:

$$d(h) = \sqrt{\frac{\alpha}{k} c_0 D^{c_1} H^{c_2 - 1} \left(1 - \frac{h}{H}\right)^{\alpha - 1}}. \tag{A.24}$$

In fact, the equivalence of A.24 and A.17 can be demonstrated algebraically with a re-parameterization. Integrating $k \cdot d^2(h)$ in A. 24 from ground to total height results in tree total volume $V_T = c_0 D^{c_1} H^{c_2}$ rather than the total volume in A.23. Thus, in taper and volume equation system constituted of A.22, A.23, and A.24, a compatible taper equation A.24 can be derived from A.22, but A.22 cannot be obtained by algebraically rearranging parameters of A.24 (integrating the taper function). Therefore, the taper and volume equations in that system are not algebraically compatible.

When $\beta = 1$, A.21 becomes $R_p = 1 - (1-p)^{\alpha}$. This ratio form has been used in earlier studies (Van Deusen 1982, Reed and Green 1984). The corresponding merchantable volume can be written as

$$V_h = c_0 D^{c_1} H^{c_2} \left\{1 - \left(1 - \frac{h}{H}\right)^{\alpha}\right\}. \tag{A.25}$$

Here, the total volume equation is

$$V_T = c_0 D^{c_1} H^{c_2}. \tag{A.26}$$

The compatible taper equation derived from A.25 has the same form as A.24. Now, A.24, A.25, and A.26 comprise an algebraically compatible taper and volume equation system.

The upper stem-diameter-based merchantable volume equations (A.1 – A.3) and the upper stem-height-based merchantable volume equations (A.22 and A.25) implicitly define the same formula of compatible taper equation (A.24). This suggests that any merchantable volume equation implicitly defines a unique taper function (Clutter 1980), but one taper function may be associated with more than one merchantable volume equations. In other words, the taper function derived from volume ratio equations (R_d, R_h, or R_p) does not guarantee the algebraic compatibility of the resultant taper and volume equation system. Zhao and Kane (2017) suggested that a volume ratio function R_p should satisfy four conditions: (I) $R_p = 0$ if $p = 0$, (II) $R_p = 1$ if $p = 1$, (III) $\partial R_p / \partial p \geq 0$ for $0 \leq p \leq 1$, and (IV) $\partial^2 R_p / \partial^2 p \leq 0$ for $0 \leq p \leq 1$. Conditions I and II are obvious, the proof of conditions III and IV is given in the appendix A of Lynch et al. (2017). These conditions could be extended for a general volume ratio function R, regardless of R_h or R_d, with respect to h: (I) $R = 0$ if $h = 0$, (II) $R = 1$ if $h = H$, (III) $\partial R / \partial h \geq 0$ for $0 \leq h \leq H$, and (IV) $\partial^2 R / \partial^2 h \leq 0$ for $0 \leq h \leq H$. The R_d and R_h are linked through a compatible taper function $d^2(h) = f(h)$. To achieve the algebraic compatibility in a taper and volume system, the volume ratio function in that system or implied in that system should satisfy all these four conditions. Because the ratio function A.20 does not meet condition I, the resultant system composed of A.22, A.23, and A.24 is not algebraically compatible. When $\beta = 1$ in A.21, the volume ratio satisfies all the four conditions, and then the resultant system composed of A. 24, A.25, and A. 26 has algebraic compatibility.

It should be emphasized that deriving R_h from R_d or deriving R_d from R_h could be possible only through a compatible equation $d^2(h) = f(h)$. Especially when $h = f^{-1}(d^2)$, an expression to predict h, could be algebraically derived from the taper equation, R_d equation could be derived from R_h (Byrne and Reed 1986), and all the derived taper equation, R_d, and R_h are compatible to the merchantable volume equation. A disadvantage of deriving taper functions by equating fits of an arbitrary R_d equation and another arbitrary R_h equation (e.g., Amateis and Burkhart 1987, Tasissa et al. 1997, Bullock and Burkhart 2003) is that the resultant taper functions cannot be guaranteed to be compatible. That is, integration of such taper functions will not necessarily be equal to the merchantable volume functions based on ratio equations in the system.

Appendix B

For a given system, three fitting methods (FM-1, FM-2, and FM-3) were ranked based on the absolute value of average errors (*E*) and standard errors of estimates (*SEE*) of all three attributes: taper, merchantable volume, and total volume in Table 4. The attributes were equally weighted. Rank one was used for the best method and three for the poorest. Table B.1 shows the sum of the ranks of the fitting methods for each system.

Table B.1. Sum of the ranks, and ranks based on the rank sum (in brackets) of the three fitting methods by the systems.

System	Description	Estimation methods		
		FM-1	FM-2	FM-3
1	Overall E for taper, merchantable, and total volume	8 (3)	5 (2)	3 (1)
	Overall SEE for taper, merchantable, and total volume	7 (3)	4 (2)	3 (1)
	Total	15 (6)	9 (2)	6 (2)
2	Overall E for taper, merchantable, and total volume	5 (1)	5 (1)	5 (1)
	Overall SEE for taper, merchantable, and total volume	5 (3)	4 (2)	3 (1)
	Total	10 (4)	9 (3)	8 (2)

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Table B.2. Sum of the ranks, and ranks based on the rank sum (in brackets) of the two systems associated with FM-3.

Description	System 1	System 2
E for diameter by DBH classes	13 (1)	23 (2)
SEE for diameter by DBH classes	21 (2)	15 (1)
E for volume by DBH classes	15 (1)	21 (2)
SEE for volume by DBH classes	15 (1)	20 (2)
E for diameter by total height classes	12 (1)	15 (2)
SEE for diameter by total height classes	14 (2)	11 (1)
E for volume by total height classes	9 (1)	18 (2)
SEE for volume by total height classes	10 (1)	17 (2)
E for diameter from ground to top	23 (2)	13 (1)
SEE for diameter from ground to top	21 (2)	15 (1)
E for volume from ground to top	14(1)	25 (2)
SEE for volume from ground to top	14 (1)	24 (2)
Total	181 (16)	217 (20)

For a given fitting method such as FM-3, the two systems were ranked based on the performance of estimation of diameters and volumes for different tree sizes and for different portions of the stem, using the average errors and standard errors of estimates in Tables 5 to 7. Each system was assigned a rank separately for every DBH class, for every total height class, and for every relative height levels for diameters and volumes. These ranks were summed for the average errors and standard errors of estimates. Table B.2 shows the sum of the ranks of the two systems associated with FM-3.

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