HONEYCOMBING THE ICOSAHEDRON AND ICOSAHEDRONING THE SPHERE[1]

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Abstract—This paper is an attempt to trace the theoretical foundations of the Forest Inventory and Analysis and Forest Health Monitoring hexagon networks. It is important in case one might desire to alter the intensity of the grid or lay down a new grid in Puerto Rico and the U.S. Virgin Islands, for instance. The network comes from tessellating an icosahedron with hexagons and projecting those hexagons to a sphere. The paper proposes a sample network for Puerto Rico and the U.S. Virgin Islands.

INTRODUCTION

Pardon the title; it is bad grammatical form to verb a noun. It is also geometrically impossible to square the circle using classical means. There will always be a few chords from the circle left over. Using the method that follows, one will find that it is impossible to completely tessellate a sphere with regular hexagons. There will be twelve pentagons left over. Historically, Forest Inventory and Analysis (FIA) plots in the American South have been laid out on a square grid or no grid at all—that is, haphazardly. Forest Health Monitoring (FHM) plots have been laid out on a hexagon network. Hexagons, squares, and triangles tile the plane (or any study area on earth). Carr and others (1999) list a set of criteria for global grid cells that argue in favor of hexagons; among other things, hexagons provide maximum area for minimum perimeter. The astrophysicist Max Tegmark (1996) listed similar criteria and built a similar grid for the sky. In geometry, squaring the circle means attempting to rearrange a circle to form a square; and in this paper, honeycombing an icosahedron is attempting to fit a honeycomb pattern on top of an icosahedron. To accomplish this task, start with an unfolded icosahedron, as shown in figure 1. Ultimately, this icosahedron shall be projected to the sphere of the earth, as shown in figure 2.

An icosahedron is a geometric solid with 20 faces, all of which are equilateral triangles. Throughout this paper, the “poles” of the icosahedron will be points at the very top and very bottom of the unfolded solid. The “cuts” will be those line segments connected to the poles, and the “ends” of the “cuts” will be the points at which the cuts join. A similar method will work with an octahedron, but the icosahedron approximates a sphere better than any other platonic solid. Tesselate each face with nine triangles. Take six of the triangles from one face to form a hexagon. There will be three triangles on each face that ultimately form parts of pentagons. The result is a solid with 32 faces, 20 of which are hexagons and 12 of which are pentagons. Geometers

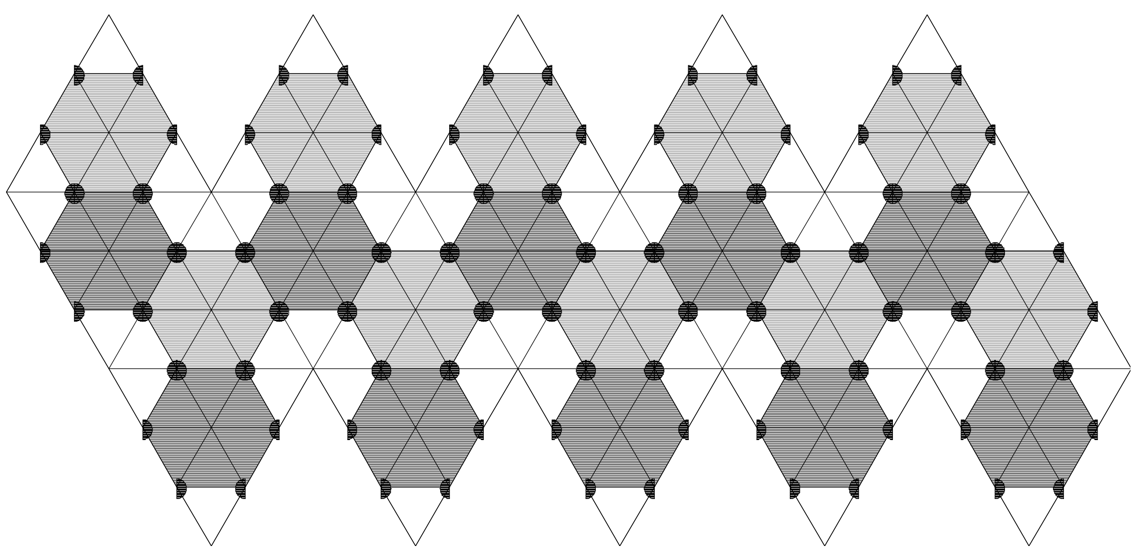

Figure 1—Icosahedron tessellated with hexagons.

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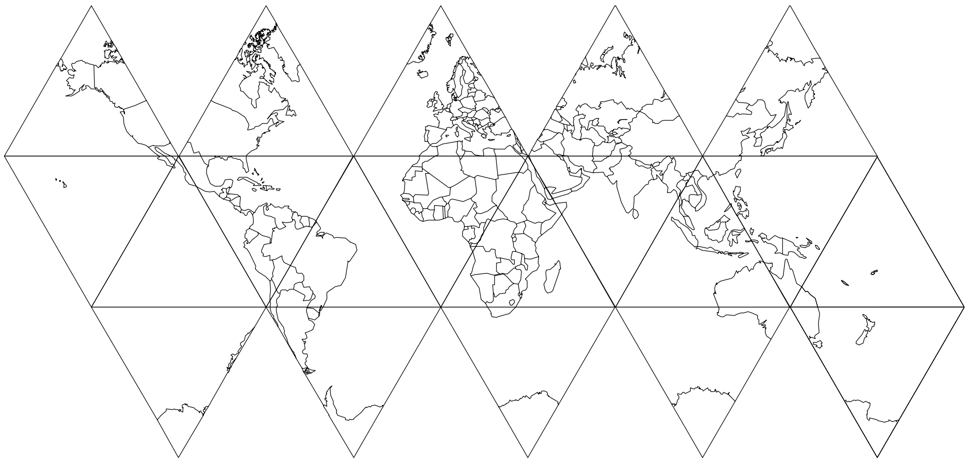

Figure 2—World map projected on an unfolded icosahedron.

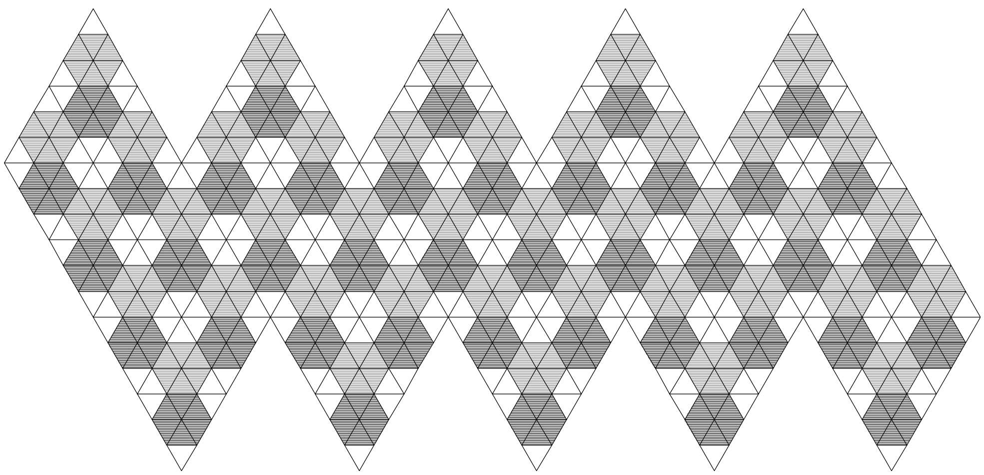

Figure 3—The second-order tessellation of the icosahedron with hexagons using the triangle orientation.

know this solid as a truncated icosahedron, athletes know it as a soccer ball; and although the reader may not be aware, virologists know it as the structure of a virus with triangulation number T = 3 (Johnson and Speir 1997). Figure 1 has sixty vertices marked with bold circles, and chemists know this structure as C_{60} (Maggio 1994).

What If More Hexagons Are Desired?

As indicated by Carr and others (1999) and Tegmark (1996), hexagons are useful because they form an ideal network by which to divide a study area into smaller areas from which to draw plots. Under the Lambert Azimuthal Equal Area projection [used in the Snyder (1992) model of this method], the default radius of the earth is 6,370,997 m. A perfect sphere of this radius has a surface area equal to 5.10×10^{14} m^2. This number is about 1.26×10^{11} ac. The icosahedron in figure 1 must be subdivided to match the scale of an FIA plot. There are 5,936 ac per FIA plot, which compared to the surface area of the earth is one part in 21 million. Each large FHM hex has 27 FIA hexes and, compared to the surface area of the earth, is 1 part in 786,000. If one wished to construct sixteenfold FIA hexes (for a more intense FHM grid), one would need a cell that is 1 part in 1,327,000. The short answer of how to create more hexagons is to create more triangles, as shown in figure 3.

Instead of the 20 hexagons and 12 pentagons in figure 1, we now have 110 hexagons and 12 pentagons. This method will work elegantly if there are $9n^2$ triangles on each face, for a total of $180n^2$ triangles. Twelve pentagons will consume 60 triangles, leaving enough triangles for $30n^2 - 10$ hexagons. Note that if n = 162, then $30n^2 - 10$ is near 786,000; if n is around 840, then $30n^2 - 10$ is around 21 million. An icosahedron that circumscribes the earth has a surface area about 14.6 percent larger than that of the sphere—one may wish to use this surface area instead of that of the earth. More exact numbers appear in table 1.

Table 1—Number and size of polygons under the triangle orientation of various resolutions

Order	Triangles	Hexagons	+	Pentagons	=	Polygons	Area of triangle	Area of hexagon
							m^2	m^2
1	180	20		12		32	2.83×10^{12}	1.70×10^{13}
2	720	110		12		122	7.08×10^{11}	4.25×10^{12}
3	1,620	260		12		272	3.15×10^{11}	1.89×10^{12}
162	4,723,920	787,310		12		787,322	1.08×10^{8}	6.48×10^{8}
840	127,008,000	21,167,990		12		21,168,002	4.02×10^{6}	2.42×10^{7}
841	127,310,580	21,218,420		12		21,218,432	4.01×10^{6}	2.41×10^{7}
842	127,613,520	21,268,910		12		21,268,922	4.00×10^{6}	2.40×10^{7}
843	127,916,820	21,319,460		12		21,319,472	3.99×10^{6}	2.39×10^{7}

HEXAGON HIERARCHIES

In a tessellation of hexagons, one hexagon is surrounded by six others, as in figure 4. If one connects the centers of the outer ring of six, one produces a hexagon three times as large as the original hexagon. One may continue in this manner to get hexagons of size 3, 12, 27, 48, . . . $3n^2$, where n = 1,2,3, . . .

If one connects opposite vertices of the ring of six as shown in figure 5, one produces a hexagon four times as large as the original hexagon. One may continue in this manner to get compositions of size 1, 4, 9, 16, 25, . . . n^2, again where n = 1,2,3, . . .

If one makes a ring of six clusters of seven hexagons around another cluster of seven hexagons, as shown in figure 6, one produces a hexagon 21 times as large as a basic hexagon. This maneuver is called a sevenfold composition. The large hexagon is seven times larger than a threefold hexagon. One may continue in this manner to get compositions of size 1, 7, 19, 37, . . . $3n^2 - 3n + 1$, where n = 1, 2, 3.

If one relaxes the constraint of requiring the vertex to be in the center hexagon of the cluster and allows the vertex to be in the center of any hexagon of the cluster, then one gets another family of compositions and decompositions, such as in figure 7.

The Chevron and Intermediate Orientations

The structures in figures 1, 2, and 3 are called the "triangle" orientation. What if one applies the threefold decomposition to the structures in figures 1, 2, and 3? Then one obtains structures as in figure 7. Structures of this sort have the chevron orientation. In this case, each face has 1.5 hexagons, plus three-fifths of a pentagon. Across 20 faces, there are 12 pentagons and 30 hexagons. In general, there are $10n^2 - 10$ hexagons and 12 pentagons for a total of $10n^2 + 2$ polygons. What resolutions yield FHM- and FIA-sized hexagons? The answer is about 280 for an FHM-sized hexagon and about an order 1,458 for an FIA-sized hexagon. Note that an order 1,458 hexagon is one twenty-seventh the size of an order 162 hexagon of the opposite orientation, and that an order 280 hexagon of this (the "chevron") orientation is 27 times as large as an order 840

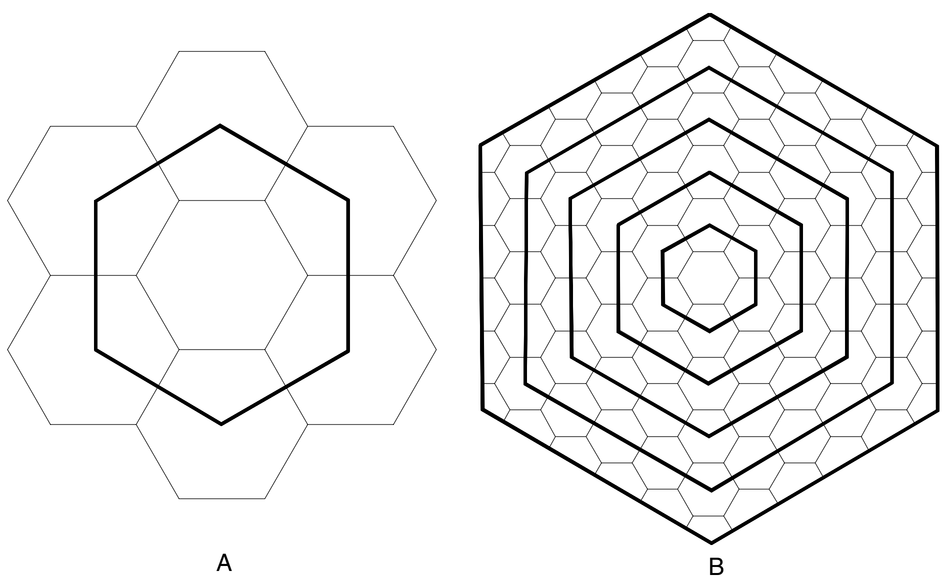

Figure 4—Illustration of (A) the threefold composition method and (B) its generalization.

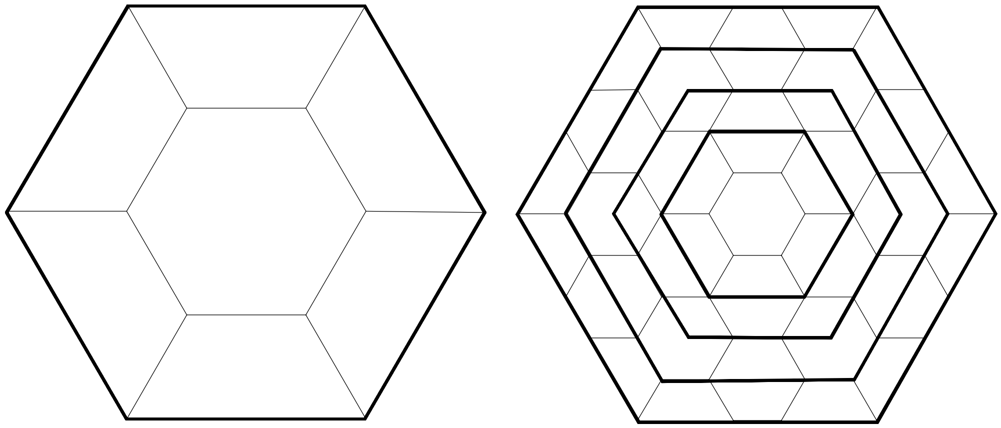

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Figure 5—Illustration of (A) the fourfold decomposition method and (B) its generalization

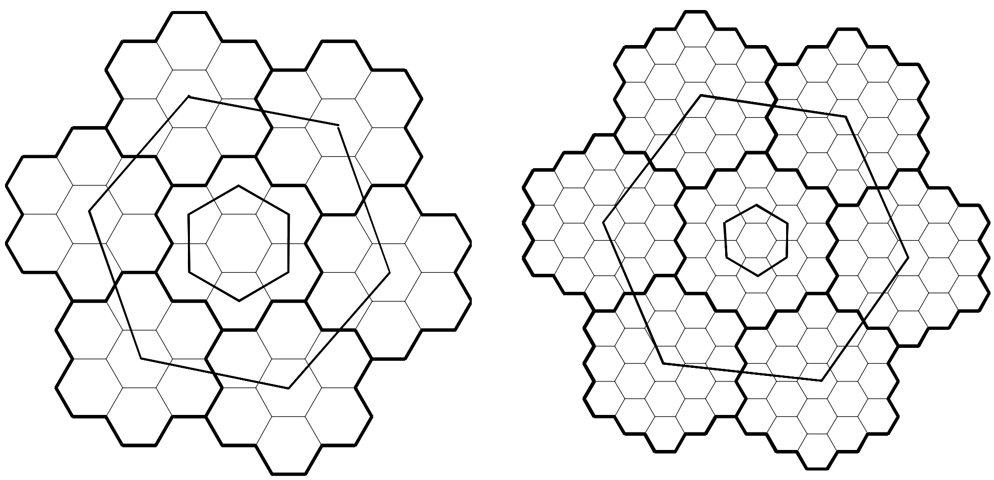

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Figure 6—Illustration of (A) the sevenfold and (B) the nineteenfold decomposition.

hexagon of the opposite (the “triangle”) orientation. More exact numbers appear in table 2.

Are the triangle and the chevron orientations the only ones possible? No, but mathematically they are the easiest. A close inspection shows that the generalized threefold decomposition is the chevron orientation. In figure 4 (A) cut the figure from the center to the vertices of the bold hexagon. What you see is the chevron orientation. In the equation $T = h^2 + hk + k^2$, substitute $h = n$ and $k = n$. The result is $T = 3n^2$. The generalized fourfold decomposition is the triangle orientation. In figure 5 (A) cut the figure from the center to the vertices of the bold hexagon. There is the triangle orientation. In the equation $T = h^2 + hk + k^2$, substitute $h = n$ and $k = 0$. What remains is $T = n^2$, and the generalized sevenfold is an intermediate case. Substitute $h = n$ and $k = n - 1$ in the equation $T = h^2 + hk + k^2$. What remains is $T = 3n^2 – 3n + 1$, the generalized sevenfold equation.

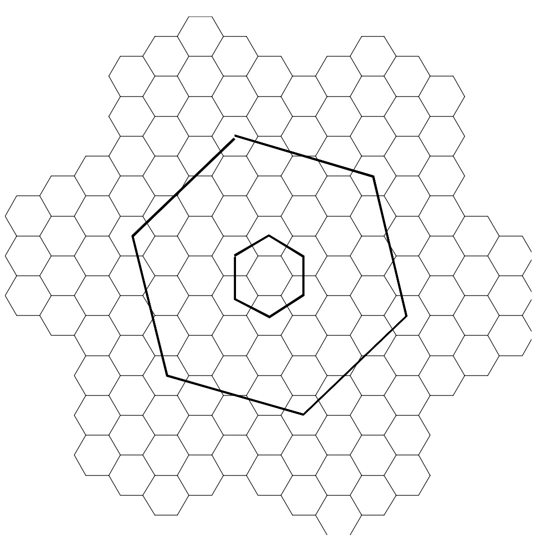

Figure 7—Thirteenfold decomposition.

Table 2—Number and size of polygons under the chevron orientation at various resolutions

Order	Triangles	Hexagons	+	Pentagons	=	Polygons	Area of triangle	Area of hexagon
							m^2	*m^2*
1	60	20		12		32	8.50×10^{12}	5.10×10^{13}
2	240	30		12		42	2.13×10^{12}	1.28×10^{13}
3	540	80		12		92	9.45×10^{11}	5.67×10^{12}
280	4,704,000	783,990		12		784,002	1.08×10^{8}	6.51×10^{8}
281	4,737,660	789,600		12		789,612	$1,08 \times 10^{8}$	6.46×10^{8}
1457	127,370,940	21,228,480		12		21,228,492	4.01×10^{6}	2.40×10^{7}
1458	127,545,840	21,257,630		12		21,257,642	4.00×10^{6}	2.40×10^{7}
1459	127,720,860	21,286,800		12		21,286,812	4.00×10^{6}	2.40×10^{7}

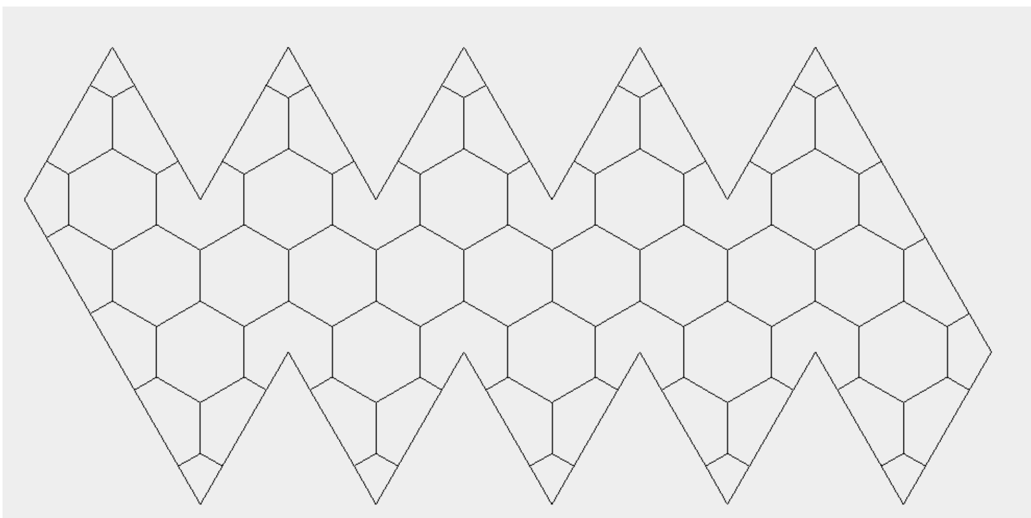

Figure 8—The chevron orientation of hexagon tessellation.

As was mentioned in the introduction, virologists refer to the “triangle” orientation as the structure of a virus with triangulation number T = 3. Triangulation numbers work in the following way:

Start with a plane of tessellated hexagons, as in figure 9. From the origin, go north by h hexagons, marking a spot at H1. Also, go northeast by h hexagons, marking a spot at H2. From H1, go north by k hexagons, marking a spot at K1. From H2, go northwest by k hexagons, marking a spot at K2. In the figure, h = 3 and k = 2. The result is an equilateral triangle with vertices at the origin, K1, and K2.

This equilateral triangle has a triangulation number $T = h^2 + hk + k^2$, or in this case, 19. An icosahedron tessellated with hexagons in this manner would have 10T – 10 hexagons, 12 pentagons, and, of course, 10T + 2 polygons. To tesselate an icosahedron with hexagons, one may continue in this fashion making sure that adjoining triangles match partial hexagons—except at the poles and the ends of the cuts, where pentagons are formed.

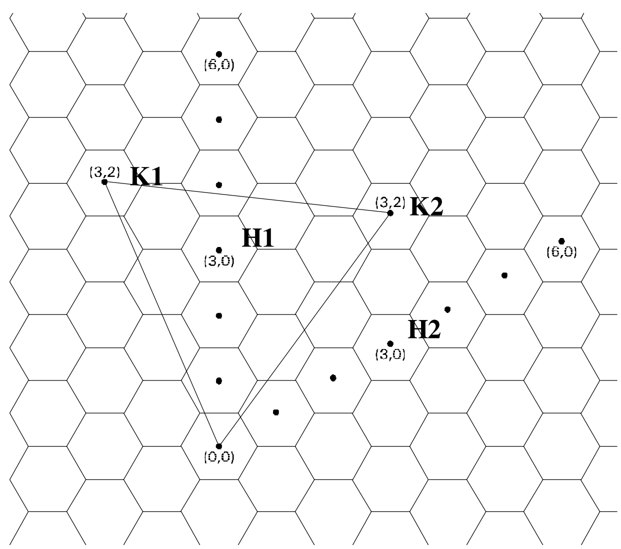

Figure 9—Method of constructing triangulation numbers.

Icosahedroning the Sphere

When the icosahedron is tessellated with hexagons in the desired way, the next problem is projecting points and lines on the icosahedron to points and lines on the sphere. This problem has baffled cartographers for decades. Among the first to offer a solution was Fisher (1943). His solution is used to this day. He used a combination of aspects of the gnomonic projection. Fisher's map appears in Dahlberg (1997) among other places.

One apparent disadvantage of Fisher's map is that the combination of aspects appears to abruptly shift from straight parallels to curved parallels. The lines of 30° N. and 30° S. look as though they appear on the map twice—once as curved lines and once as straight lines. Thus, directions could be ambiguous and certain points on the earth might appear on the earth in two different places. Snyder used a variation of the Lambert Azimuthal Equal Area projection, and Buckminster Fuller devised a method as well (Pitre 2000).

If you want to map the icosahedron to the sphere, it is desirable that the points on the main triangles match. That way, no point on the earth appears on the map more than once and directions are unambiguous. To make computations easier, you may want to start with the North Pole at the top apex and the South Pole at the bottom apex. Then, split the sphere into five parts with each cut 72° from the next.

One tempting solution is to map parallels on the earth to straight lines in the triangle orientation. Doing so, one can see that 5 triangles (out of 20) join at each of the poles. One can show that one-quarter of a sphere's area is above 30° N., and one-quarter below 30° S. So simply map straight lines in the triangle to parallels on the earth. Meridians sweep out equal areas of the triangle. One flaw in that method is that when half-hexagons are joined from large triangles, the resultant figures look more like pentagons than hexagons.

The next attempt might be to map straight lines on the triangle to great circles on the earth. The cuts are still mapped to meridians (which are great circles). Straight lines are mapped to great circles, which follow such equations as atan(c•sinθ). Munem and Foulis (1984) give the area of a sphere as:

$$\int_{0}^{\pi}\int_{0}^{2\cdot\pi} R^{2}\cdot\sin(f)\,dt\,df \qquad (1)$$

where

R = the radius,
t = longitude, and
f = colatitude.

One may adapt this equation to:

$$\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^{2}\cdot\cos(f)\,df\,dt \qquad (2)$$

where

f = latitude.

If one wants the area bounded by any 72° wedge, the North Pole, and a particular great circle, one gets:

$$\int_{\frac{\pi}{2}-\frac{\pi}{5}}^{\frac{\pi}{2}+\frac{\pi}{5}}\int_{\operatorname{atan}(c\cdot\sin(t))}^{\frac{\pi}{2}} R^{2}\cdot\cos(f)\,df\,dt \qquad (3)$$

Integrate this to get:

$$\frac{2}{5}\cdot\pi\cdot R^{2} - 2\cdot\operatorname{asin}\left(\frac{1}{4}\cdot\sqrt{2}\cdot\sqrt{5-\sqrt{5}}\cdot\frac{c}{\sqrt{1+c^{2}}}\right)\cdot R^{2} \qquad (4)$$

If one wants one-twentieth of the total area of the sphere, one sets this result equal to $\pi R^2/5$, and then solves for c, which turns out to be equal to 0.618034. If one substitutes this value into atan(k•sinθ), one sees that the great circle traces a route from 26.565 N. at the endpoints (the 54^{th} and 126^{th} meridians) to an apex of 31.717 N, at the 90^{th} meridian.

If one wants a mathematically simple method, one can just project any point (x,y,z) on the triangular face of the icosahedron to a point (X,Y,Z) on a sphere of radius R in the following way:

$$r = x^2 + y^2 + z^2 \qquad (5)$$

$$X = (R/r)\bullet x \qquad (5a)$$
$$Y = (R/r)\bullet y \qquad (5b)$$
$$Z = (R/r)\bullet z \qquad (5c)$$

This method leads to hexagons of roughly equal size. Exactly equal sized hexagons are desirable, but not crucial to planning a forest inventory (Snyder 1992).

One could adapt the gnomonic method to a more equal area projection by observing that in the triangle orientation, the first row has one triangle, the second three, the third five, . . . and k^2 triangles up to and including the k^{th} row. If there are n^2 total triangles on the face of a major triangle, one may set expression (4) equal to $(\pi R^2/5)\bullet(k^2/n^2)$ and solve for c. Also, meridians would not be equally spaced on the triangle. Observe that:

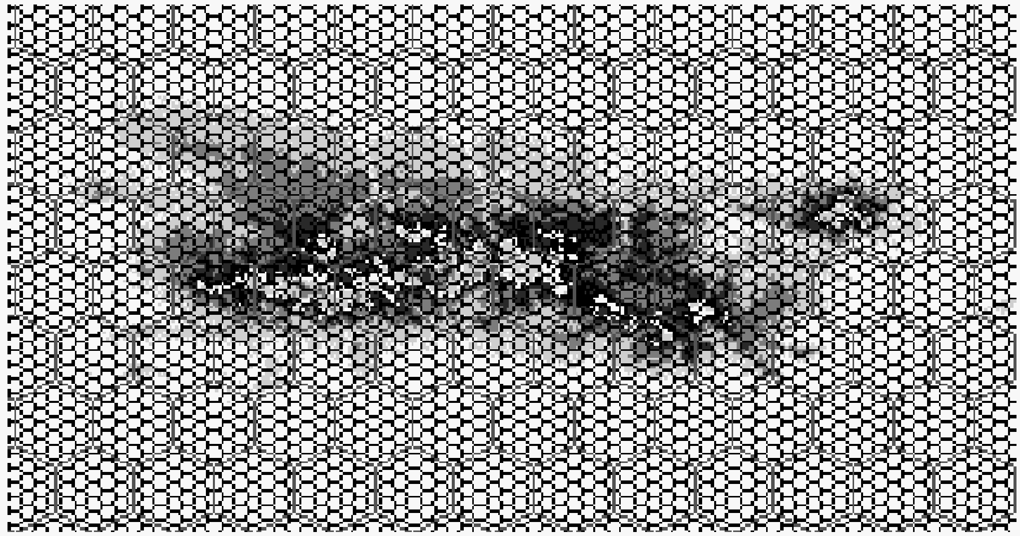

Figure 10—Hexagon grid for Puerto Rico and the U.S. Virgin Islands

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+k} \int_{\operatorname{atan}(c \cdot \sin(t))}^{\frac{\pi}{2}} R^2 \cdot \cos(f)\, df\, dt \qquad (6)$$

is equal to:

$$R^2 \cdot k - \operatorname{asin}\left(\sin(k) \cdot \frac{c}{\sqrt{1+c^2}}\right) \cdot R^2 \qquad (7)$$

which in general is not linear in k. At latitudes near Puerto Rico and the U.S. Virgin Islands, it makes little difference whether one maps straight lines on the icosahedron to parallels on the earth, great circles on the earth, or loxodromes (lines of constant direction) on the earth. However, at latitudes near Alaska, it does make a difference.

In conclusion, figure 10 is a proposed grid system for Puerto Rico and the Virgin Islands. This grid may need to be altered in order to accommodate Snyder's (1992) assumptions and starting points as well as for the particular needs of the FIA and FHM programs in the Caribbean. At this writing, various resolutions of the grid are being explored. In the conterminous United States, FIA-sized hexagons have been grouped into sets of 27 for the purpose of constructing traditional FHM-sized hexagons. One could also overlay another network of sixteenfold hexagons.

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