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THEORY OF GAMES AND APPLICATIONS IN FORESTRY

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Theory of Games and Applications in Forestry

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I. Introduction and Theoretical Aspects

I.0. Game Theory in Forestry

The theory of games of strategy derives from von Neumann and Morgenstern (1944). Their book was greeted with considerable enthusiasm by military strategists, economists, and others. The elegance of their development of the theory is much admired. Still, substantive applications within forestry and other natural resources fields have been rare. The root cause of this paucity of applications is not clear. Surely the theory is little known to forest managers and decisionmakers. All this suggests that an exposition of game theory for natural resources managers could encourage application of the techniques in solving real problems. Hence this paper. Aside from that, the theory has an intrinsic beauty and study of such abstractions, as of mathematics generally, inculcates lofty habits of mind.

There are very obvious limitations to the applicability of game theory. Thus, two-person, constant-sum games assume two knowledgeable and selfish adversaries with opposed interests. Clearly, the adversaries can be two people, but need not be. They can as well be two agencies, two companies, two coalitions, two armies. All that is required is that individuals in each group share, so far as the game being played is concerned, a common interest.

The interests must be strictly opposed. One player's gain is the other's loss. Each is assumed knowledgeable of all options and their consequences. Each is expected to try to maximize his own gain, implying he seeks to maximize his opponent's loss. Such situations must be rare within a single organization with a united purpose. Thus, within a company or agency managing land for profit or public good, it appears that opportunities for strict applicability of this theory as a guide to rational behavior are indeed rare. Even between companies within an industry or between departments within an agency, compelling examples of complete antagonism do not readily come to mind in profuse abundance.

Still, there are instances of conflicting situations that can be described fundamentally as a game of strategy between two or more adversaries. The insight provided by the game theoretic formulation and solution can be a helpful guide to choosing, predicting, or understanding rational behavior in such circumstances. A few examples that may be suggestive of many are given in **§II.** Those are necessarily much simplified in order to treat them in the space allotted and so that the techniques of problem solving may not be lost in the complexities of more realistic situations.

In order that the theory may not seem trival, we now attempt to show that the logic of game theoretic abstractions has profound implications for large problems (as well as delightful implications for small ones). To do so we extract freely from Garrett Hardin (1968). Hardin's thesis was that the world population problem has no technical solution but requires rather a fundamental extension of morality. Some of his analyses are used out of that context--since they apply equally to the one we consider.

With regard to some of its attributes, the forests of a nation are often conceived as a commons. Their beauty and protective influence over water and wildlife resources are thought of as a natural resource. Concerned citizens frequently express an interest in the conservation of these resources and sometimes object to management measures implemented to achieve the immediate objectives of the landowner.

In Hardin's words:

"The tragedy of the commons develops in this way. Picture a pasture open to all. It is to be expected that each herdsman will try to keep as many cattle as possible on the commons. Such an arrangement may work reasonably satisfactorily for centuries because tribal wars, poaching, and disease keep the numbers of both man and beast well below the carrying capacity of the land. Finally, however, comes the day of reckoning, that is, the day when the long-desired goal of social stability becomes a reality. At this point, the inherent logic of the commons remorselessly generates tragedy.

"As a rational being, each herdsman seeks to maximize his gain. Explicitly or implicitly, more or less consciously, he asks, 'What is the utility to me of adding one more animal to my herd?' This utility has one negative and one positive component.

(1) The positive component is a function of the increment of one animal. Since the herdsman receives all the proceeds from the sale of the additional animal, the positive utility is nearly +1.

(2) The negative component is a function of the additional overgrazing created by one more animal. Since, however, the effects of overgrazing are shared by all the herdsmen, the negative utility of any particular decision-making herdsman is only a fraction of -1.

"Adding together the component partial utilities, the rational herdsman concludes that the only sensible course for him to pursue is to add another animal to his herd. And another; and another.... But this is the conclusion reached by each and every rational herdsman sharing a commons. Therein is the tragedy. Each man is locked into a system that compels him to increase his herd without limit--in a world that is limited. Ruin is the destination toward which all men rush, each pursuing his own best interest in a society that believes in the freedom of the commons. Freedom in a commons brings ruin to all."

Do these concepts of a constant-sum game, and the tragic implications of dominated strategies so graphically described, have any analogous implications in the complex problems of formulation of forest resources management policy? Hardin himself gives both a specific and general illustration that establishes the affirmative.

"The National Parks present another instance of the working out of the tragedy of the commons. At present, they are open to all, without limit. The parks themselves are limited in extent--there is only one Yosemite Valley--whereas population seems to grow without limit. The values that visitors seek in the parks are steadily eroded. Plainly, we must soon cease to treat the parks as commons or they will be of no value to anyone.

"What shall we do? We have several options. We might sell them off as private property. We might keep them as public property, but allocate the right to enter them. The allocation might be on the basis of wealth, by the use of an auction system. It might be on the basis of merit, as defined by some agreed-upon standards. It might be by lottery. Or it might be on a first-come, first-served basis, administered to long queues. These, I think, are all the reasonable possibilities. They are all objectionable. But we must choose--or acquiesce in the destruction of the commons that we call our National Parks... In a reverse way, the tragedy of the commons reappears in problems of pollution. Here it is not a question of taking something out of the commons, but of putting something in--sewage, or chemical, radioactive, and heat wastes into water; noxious and dangerous fumes into the air; and distracting and unpleasant advertising signs into the line of sight. The calculations of utility are much the same as before. The rational man finds that his share of the cost of the wastes he discharges into the commons is less than the cost of purifying his wastes before releasing them. Since this is true for everyone, we are locked into a system of 'fouling our own nest,' so long as we behave only as independent, rational, free-enterprisers."

Hardin concludes that the solution to producing temperance in the use of a commons resides in mutual coercion mutually agreed upon. Importantly, he remarks:

"It is worth noting that the mortality of an act cannot be determined from a photograph. One does not know whether a man killing an elephant or setting fire to the grassland is harming others until one knows the total system in which his act appears. 'One picture is worth a thousand words,' said an ancient Chinese; but it may take 10,000 words to validate it. It is as tempting to ecologists as it is to reformers in general to try to persuade others by way of the photographic shortcut. But the essence of an argument cannot be photographed: it must be presented rationally--in words."

The compelling logic of the foregoing analysis leads to the conclusion that policies to protect forest resources will best rely on mutual coercion mutually conceived from reliable evidence concerning the functioning of the total forest ecosystem. Again quoting Hardin,

"reaching an acceptable and stable solution will surely require more than one generation of hard analytical work--and much persuasion."

Persuasion and agreement depend on understanding the analytical work--including the tools of the analysis. Occasionally the appropriate analytical tool is the theory of games of strategy.

I.1. Zero-Sum, Two-Person Games

The following abstraction taken as a definition of a game of strategy will be sufficiently general for most purposes. One of two players is required to choose one among the finite set of strategies

$$A = \{ a_1, a_2, \ldots, a_i, \ldots, a_m \}.$$

The second player is required to choose one strategy from the finite set

$$B = \{ b_1, b_2, \ldots, b_j, \ldots, b_n \}.$$

Each player is aware of all of his own and all of his opponent's alternatives, but must choose his own strategy without benefit of knowing his opponent's choice. Subsequent to a choice of strategy, say a_i, by Player I, and a choice of strategy, say b_j, by Player II, Player II is required to pay Player I an amount ℓ_{ij} (possibly negative). Such games are called two-person games for obvious reasons and zero-sum games since the sum of the winnings of the two players is zero; there is no "house." Clearly, zero-sum, two-person games are characterized by their payoff matrix which is known to both players.

	Player II Strategies					
	b_1	b_2	...	b_j	...	b_n
a_1	ℓ_{11}	ℓ_{12}	...	ℓ_{1j}	...	ℓ_{1n}
a_2	ℓ_{21}	ℓ_{22}	...	ℓ_{2j}	...	ℓ_{2n}
...	...	...	...	...	...	...
a_i	ℓ_{i1}	ℓ_{i2}	...	ℓ_{ij}	...	ℓ_{in}
...	...	...	...	...	...	...
a_m	ℓ_{m1}	ℓ_{m2}	...	ℓ_{mj}	...	ℓ_{mn}

It is obvious that Player II would like to choose a strategy so that the payoff (which is his loss) is as small as possible. But, the payoff depends on the strategy chosen by Player I. So, the question arises as to whether Player II has an optimum strategy. And conversely for Player I, who wishes to make the payoff (which is his gain) as large as possible.

I.2. Strictly Determined Games

In the payoff matrix for certain games there is a payoff, ℓ_{ij} say, simultaneously not larger than any payoff in the same row, and not smaller than any payoff in the same column. Such payoffs are said to be a saddlepoint, the game is said to be strictly determined, and ℓ_{ij} is called the value of the game.

Consider the implications to the two players should the payoff matrix in **§I.1** contain a saddlepoint at ℓ_{ij}. If Player I then chooses the strategy a_i, he guarantees himself a payoff (gain) of at least ℓ_{ij}--no matter what strategy Player II chooses. Conversely, if Player II chooses the strategy b_j, he guarantees himself a payoff (loss) of no more than ℓ_{ij}. In this situation, it is clear that neither player may, by any alternate choice of strategy, hope to obtain more favorable results. Player I cannot do better than choose a_i, which is called a maximim pure strategy and is optimum for him. Player II cannot do better than choose b_j, which is called a minimax pure strategy and is optimum for him. The payoff is then $\ell_{ij} = v$, say the value of the game.

I.3. Inadmissible Strategies

In some payoff matrices, a systematic search for optimum strategies is facilitated by deleting certain rows and/or columns. In the payoff matrix of **§I.1**, suppose the first two columns possess the property:

$$\ell_{i1} \geqslant \ell_{i2}, \text{ all } i = 1, 2, \ldots, m, \text{ and}$$

$$\ell_{i1} > \ell_{i2}, \text{ some } i = 1, 2, \ldots, m,$$

i.e., in each row the payoff in column 1 is always as large as the payoff in column 2--and sometimes larger. In that case, choosing b_2 is always as good as choosing b_1 for Player II, and sometimes (for countering some choices of his opponent) it is better. Player II can have no reason ever to prefer strategy b_1 to b_2, and b_1 is said to be dominated by b_2. Any strategy of either player so dominated by an alternative strategy is said to be inadmissible and may be deleted from further consideration.

I.4. Computations for Strictly Determined Games

The payoff matrix of **§I.1** may be systematically examined to determine whether it has a saddlepoint or not by appending a column (composed of the row minima) and a row (composed of the column maxima) as follows.

		Player II Strategies				
		b_1	b_2	$\cdots$	b_n	
Player I Strategies	a_1	ℓ_{11}	ℓ_{12}	$\cdots$	ℓ_{1n}	$\min_j \ell_{1j}$
	a_2	ℓ_{21}	ℓ_{22}	$\cdots$	ℓ_{2n}	$\min_j \ell_{2j}$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	a_m	ℓ_{m1}	ℓ_{m2}	$\cdots$	ℓ_{mn}	$\min_j \ell_{mj}$
		$\max_i \ell_{i1}$	$\max_i \ell_{i2}$	$\cdots$	$\max_i \ell_{in}$	

If the maximum of the elements in the appended column (i.e., the maximum of the row minima) is equal to the minimum of the appended row (i.e., the minimum of the column maxima), then the game is strictly determined (i.e., the payoff matrix has a saddlepoint). Moreover, any strategy of Player I that maximizes the row minima is a maximin strategy and an optimum strategy for him. Conversely, any strategy of Player II that minimizes the column maxima is a minimax strategy and an optimum strategy for him. Finally, the value of the game is

$$v = \max_i \min_j \ell_{ij} = \min_j \max_i \ell_{ij} .$$

It may be noted that a payoff matrix may have more than one saddlepoint. The payoff is always v at each of them. And, when either player has more than one optimum strategy, he will be indifferent between them since each guarantees a payoff of v.

I.5. Mixed Strategies

We have completed our theoretical description of strictly determined games of strategy. Henceforth we are particularly interested in games with payoff matrices where

$$\max_i \min_j \ell_{ij} < \min_j \max_i \ell_{ij}$$

(the opposite inequality is arithmetically impossible). From **§1.4** it is obvious that Player II always has a pure strategy that ensures his loss will be no greater than the right-hand side of the last inequality. Analogously, Player I has a pure strategy that ensures his gain will be no less than the left-hand side. In **§1.4** these two quantities were equal, and discovery of such strategies constituted a solution of the game.

Now the disparity between the two sides of the last inequality may be viewed opportunistically by both players--for it suggests the possibility of increased gain (reduced loss) over that guaranteed by the maximim (minimax) pure strategy. The fundamental theorem of games addresses this possibility and provides a definitive solution for each player. It employs the concepts of mixed strategies for the players and expected payoff in the repetitive play of the same game.

Consider then that Player I can elect to choose among his pure strategies, a_1, with the aid of a random device. Thus, with such a device, he might choose strategy a_1 with probability p_1; a_2 with probability p_2, ..., and strategy a_m with probability p_m where:

$$0 \leqslant p_i, \; i = 1, 2, \cdots, m; \text{ and } 1 = \sum_{i=1}^{m} p_i .$$

(For example, if in each play of a certain game, Player I has only two pure strategies, a_1 and a_2, and he chooses a_1 if the flipping of a (fair) coin shows heads, and he chooses a_2 if the coin shows tails, then $m = 2$ and $p_1 = p_2 = 1/2$.) Choice of a particular set of probabilities

$$\underline{p} = [\; p_1 \;\; p_2 \;\; \cdots \;\; p_m \;]$$

constitues choice of a mixed strategy by Player I. The set of all mixed strategies available to Player I is the set of all 1xm vectors, such as $\underline{p}$ above, where the elements of $\underline{p}$ are nonnegative and sum to one.

Analogously, Player II could choose among his pure strategies, b_1, b_2, $\cdots$, b_n, with probabilities

$$\underline{q} = [\; q_1 \; q_2 \; \cdots \; q_n \;] ,$$

i.e., he could choose a mixed strategy by electing one among all 1xn vectors whose elements are nonnegative and sum to one.

I.6. Expected Payoff

Given a particular choice of a mixed strategy, $\underline{p}$, by Player I, and a particular choice of a mixed strategy, $\underline{q}$, by Player II, it is a simple computation to find the expected payoff in a long series of repetitions of a zero-sum, two-person game. For the expected payoff is the sum over all prospective payoffs of the product of the payoff and its probability of occurrence. Clearly the probability of any payoff in the payoff matrix, say ℓ_{ij}, is

$$\Pr(\ell_{ij}) = p_i \; q_j$$

since ℓ_{ij} is realized if and only if Player I chooses his strategy a_i (which he does with probability p_i), and Player II chooses his strategy b_j (which he does with probability q_j), and since (**§I.1**) choice by each player is made in the absence of knowledge, and therefore independently, of his opponent's choice. Thus, given $\underline{p}$ and $\underline{q}$, the expected payoff is

$$\sum_i \sum_j \ell_{ij} \Pr(\ell_{ij}) = \sum_i \sum_j \ell_{ij} \, p_i q_j = \underline{p} \; L \; \underline{q}'$$

where $\underline{q}'_{nx1}$ is the matrix transpose of $\underline{q}_{1xn}$, $L = L_{mxn}$ is the payoff matrix, and $\underline{p}$ L $\underline{q}$, is simply the matrix product

$$\underline{p} \, L \, \underline{q}' = [\; p_1 \; p_2 \; \cdots \; p_m \;] \begin{bmatrix} \ell_{11} & \ell_{12} & \cdots & \ell_{1n} \\ \ell_{21} & \ell_{22} & \cdots & \ell_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \ell_{m1} & \ell_{m2} & \cdots & \ell_{mn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \cdots \\ q_n \end{bmatrix}$$

which is a scalar, i.e., one number.

I.7. The Fundamental Theorem

The fundamental theorem of games of strategy given by John von Neumann established that for every payoff matrix L

$$\max_{\underline{p}} \ \min_{\underline{q}} \ \underline{p} \ L \ \underline{q}'$$

and

$$\min_{\underline{q}} \ \max_{\underline{p}} \ \underline{p} \ L \ \underline{q}'$$

both exist and they are equal. Their common value, say V, is called the value of the game. Stated another way, the theorem establishes that in any zero-sum, two-person game Player I has a strategy, $\underline{p}^*$, among his strategies $\underline{p}$, and Player II has a strategy, $\underline{q}^*$, among his strategies $\underline{q}$, such that

$$\underline{p} \ L \ \underline{q}^{*\prime} \leqslant V = \underline{p}^* \ L \ \underline{q}^{*\prime} \leqslant \underline{p}^* \ L \ \underline{q}'.$$

Thus, from the left inequality, Player II has a strategy, $\underline{q}^*$, that guarantees that his expected loss will not exceed V no matter what strategy Player I chooses. And, from the right inequality, Player I has a strategy, $\underline{p}^*$, that guarantees that his expected gain will be at least V no matter what strategy Player II chooses. Since neither player, by alternative choice of strategy, can improve his prospects, each may as well choose $\underline{p}^*$ and $\underline{q}^*$, respectively, so as to achieve V in the long run: $\underline{p}^*$ is a maximin mixed strategy and optimum for Player I; $\underline{q}^*$ is a minimax mixed strategy and optimum for Player II.

It should be observed that the choice of a pure strategy, a_i, by Player I, i.e., the certain choice of a_i can be described as the choice of the mixed strategy with $p_i = 1$ and $p_k = 0$, where $k \neq i$. And, similarly, the choice of the pure strategy, b_j, by Player II can be described as the choice of the mixed strategy where all elements are zero except for one in the j^{th} position.

When the payoff matrix has a saddle-point, the maximin pure strategy satisfies the defining property of $\underline{p}^*$, the optimum mixed strategy for Player I, and the minimax pure strategy satisfies the defining property of $\underline{q}^*$, the optimum mixed strategy for Player II.

I.8. Mention of Other Games

Extensions in several directions of the theory of the preceding sections have been attempted. Some of them will now be briefly reviewed.

Games involving three persons, four persons, and in general many persons have been formally considered. Here we will confine ourselves to quoting a single paragraph from Dorfman and others (1958):

"The theory of many-person games in the hands of von Neumann and Morgenstern is essentially a theory of coalitions, their formation and revision. The underlying idea is that two persons in such a situation cannot do worse by acting jointly than by acting severally, and may do better. Thus a many-person game tends to be reduced to a two-'person' game in which each 'person' is a coalition. The problems then become: which coalitions will form and how will the winnings be divided among the members of the coalition? To pursue the answers proposed for these questions would lead us into a specialized discussion, and since these answers are not very satisfactory we refrain."

Extending the results of zero-sum games to constant-sum games is easy. Indeed, in some elementary expositions the basic theory is presented on the basis of the assumption that the sum of the winnings of the two players is always a constant, say ℓ. We have taken ℓ to be zero for convenience.

Further extension to non-constant-sum games is much more difficult. When one player's gain is not necessarily the other's loss, there is the possibility of increased gain by both players through collusion and cooperation. Unfortunately, as McKinsey (1952) observes:

"Despite the great importance of general games for the social sciences, there is not available so far any treatment of such games which can be regarded as even reasonably satisfactory."

I.9. Decision Theory

One application of the theory of zero-sum, two-person games envisions a game of strategy between a decisionmaker and "nature." Some modification of the preceding arguments is required, and there is considerble controversy. Moreover, even a cursory review of the subject would require a treatment comparable in size to that given here to games of conflict between selfish opponents. Still, this important subject must be mentioned.

So, we now consider briefly that the abstraction of **§I.1** is sometimes used to describe the situation of a decisionmaker (Player II) "in the real world" confronted with choosing among decisions or actions, $b_1, b_2, \cdots, b_n$. In the natural resources field, the sets of actions confronting various decisionmakers are as diverse as one can imagine. They may deal with business, forest management, personnel, silviculture, engineering, and selection, its maintenance or deployment of equipment. The list goes on. Typically, choice of a particular decision commits one for the future. Even if it doesn't, the optimum choice of action is typically not evident due to uncertainties about markets, economic conditions, natural events such as weather to be encountered, action or state of biological agents, etc. In any event, it is possible to envision the applicability of the materials of **§I.1** to some of these situations by taking Player I to be, for lack of a better descriptor, "nature" and the strategies of "nature" to be the various events or conditions that the decisionmaker may confront. The payoff, ℓ_{ij}, becomes the cost (possibly negative, indicating a net benefit) of taking action, b_j, should condition a_i materalize. Clearly, Player II wishes to choose an action, b_j, to minimize his costs (maximize his benefits). But, in the formulation of **§I.1**, as in real life, uncertainty about the condition, a_i, that may prevail creates uncertainty about the desirable action.

Now, a disparity. Previously, Player I was considered to be an intelligent, deliberative opponent, informed of the payoff matrix, and expected to choose a strategy, a_i, to maximize Player II's loss. "Nature," whether denoting next month's weather, the biological agents at work in a forest, or general economic conditions expected for the next quarter, can hardly be construed as so perverse. Still, on occasion, the minimax solution given by game theory is recommended for decisionmakers who simply wish to guard against the worst contingency.

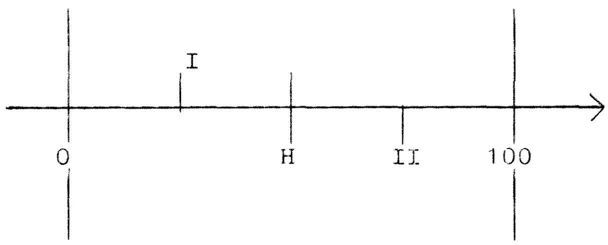
More often, an alternative base on prognosis is recommended. For example, if weather is the adversary, and potential conditions are enumerated, then something of the probability of their occurrence must be known--perhaps from climatic records. If the period of time of concern to the decisionmaker is near and short, perhaps the historical frequency of conditions may be modified by a weather forecast. Analogously, enough of the health of a forest may be known to specify the likelihood of various biological conditions, and there are forecasts of future economic conditions. What is usually recommended to the decisionmaker equipped with information on the likelihood of various conditions is that he determine his expected loss under each of his alternatives and choose that one minimizing his expected loss given his information. That is, a Bayesian strategy, is usually recommended in such formulations as better justified than a minimax one.

For the reader interested in decision theory, several references in **§IV** are selected from a voluminous literature.

II. Procedural Aspects and Illustrative Applications

II.1. Locations for Two Companies in a Forest

Suppose a single railroad traverses a homogeneously forested region. Within the forest, distances along the railway from one boundary to the other are scaled from zero to 100. Suppose two forest products companies intend to enter this market by constructing a single rail siding each where wood may be scaled, purchased, and loaded for transport to their respective mills. They must locate at two points, such as I and II, in the following figure.



If we construct a point, H, halfway between I and II and assume each company will get all of the wood marketed from its side of point H, where should the two companies locate their siding?

This problem fits very neatly into the format of a zero-sum, two-person game (**§I.1**) if it's assumed for simplicity that the companies can locate only at a finite number of points along the railway, say at 0, 20, 40, 50, 60, 80, and 100. Suppose further that the companies will split the market 50-50 if they choose the same location. The payoff matrix is shown in the table below. The losses shown in the table are the proportions of the market yielded by Company II to Company I, depending on the two locations chosen. Clearly, Company II wishes to choose a location (strategy) to minimize such a payoff. Company I wishes to maximize it.

To ascertain whether this payoff matrix has a saddlepoint (**§I.1**), the minimum loss in each row is appended on the right of the payoff matrix (**§I.4**). Analogously, the maximum loss in each column is appended to the bottom. Finally, it is observed that the maximum of the row minima is .50, as is the minimum of the column maxima, i.e., the two are equal and this payoff matrix has a saddlepoint. Moreover, the optimum location (maximum strategy) for Company I is in the center of the forest (at location 50). The optimum location (minimax strategy) for Company II is also in the center (at location 50).

		Location of Company II							
		0	20	40	50	60	80	100	Row minima
Location of Company I	0	.50	.10	.20	.25	.30	.40	.50	.10
	20	.90	.50	.30	.35	.40	.50	.60	.30
	40	.80	.70	.50	.45	.50	.60	.70	.45
	50	.75	.65	.55	.50	.55	.65	.75	.50 ← maximum
	60	.70	.60	.50	.45	.50	.70	.80	.45
	80	.60	.50	.40	.35	.30	.50	.90	.30
	100	.50	.40	.30	.25	.20	.10	.50	.10
Column maxima		.90	.70	.55	.50 ↑ minimum	.55	.70	.90	

When both companies locate in the center, they share the market 50-50 (.50 is the value of the game). The payoff corresponding to the two optimum strategies exhibits the characteristic property of a saddlepoint--it is smallest in the row and largest in the column containing it (§I.2).

Notice that locating one company at any point other than the center yields more than half of the market to the competitor who locates in the center.

II.2. When to Patrol

Consider next a problem that confronts, in more or less complex form, many forest managers whose jobs include an element of law enforcement, i.e., whether to patrol an area susceptible to unlawful entry. Problems of this type arise in many forms due to the susceptibility of forest land to timber trespass, poaching, arson, etc.

Suppose a company forester is alerted that his lands are the intended target of an arsonist. During any particular susceptible period (e.g., overnight) the arsonist may either attempt to set fire or he may not (i.e., he may stay home). The forester's alternative strategies may be to patrol or refrain. Suppose the payoff matrix is

Action of Arsonist	Action of Forester: Patrol	Don't patrol	Row minima
Burn	-100	10	-100
Don't burn	1	0	0
Column maxima	1	10	

The numerical values of these payoffs are debatable--as is the assumption of zero-sum payoffs. But, the rationale could be something like this. If neither protagonist acts, the (mutual) result is the status quo--indicted by zero payoff. If the patrol is employed in the absence of the arsonist, the company incurs a cost of 1 man-day of unnecessary labor. Alternatively, if the arsonist strikes in the absence of a patrol, the company incurs 10 man-days' labor in suppressing the fire. Both of these labor costs are assumed to accrue in satisfaction to the arsonist as payment for a real or imagined injustice by the company. Finally, if both protagonists act, the arsonist is caught and fined and/or imprisoned at considerable personal loss. The company correspondingly gains from the relief from future patrols and suppression from the arsonist apprehended and perhaps from others who might entertain like ambitions for revenge or mischief in the absence of an example of the potential consequences.

This simple game does not possess a saddlepoint since

$$\max_i \min_j \ell_{ij} = 0 < 1 = \min_j \max_i \ell_{ij},$$

i.e., the maximum of the row minima is less than the minimum of the column maxima. Hence, the optimum strategy for at least one of the players involves a mixture of his pure strategies, and the value of this game has not been determined.

A general algebraic solution for 2x2 games with payoff matrices

$$\begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

without saddlepoints is available. In such games, Player I's optimum mixed strategy is to choose his first pure strategy with probability p and his second with probability 1-p where

$$p = \frac{\ell_{22} - \ell_{21}}{\ell_{11} - \ell_{12} - \ell_{21} + \ell_{22}}.$$

Analogously, Player II's optimum mixed strategy is to choose his first and second pure strategies with probabilities q and 1-q, respectively, where

$$q = \frac{\ell_{22} - \ell_{12}}{\ell_{11} - \ell_{12} - \ell_{21} + \ell_{22}}.$$

Finally, the value of the game is

$$V = \frac{\ell_{11} \cdot \ell_{22} - \ell_{12} \cdot \ell_{21}}{\ell_{11} - \ell_{12} - \ell_{21} + \ell_{22}}.$$

Using the numerical payoffs given at the beginning of this section and the relevant formulas, it is easy to find that

(i) the optimum strategy for the arsonist is to burn the woods with probability $p \approx 1/111 \approx .009$ and to stay home with probability $1-p = 110/111 \approx .991$,

(ii) the optimum strategy for the forester is to patrol with probability $q = 10/111 \approx .09$ and to refrain with probability $1-q = 101/111 \approx .91$, and

(iii) the value of the game is $V = 10/111$.

II.3. A Graphical Method for Finding an Optimal Strategy for Any Player With Two Strategies

Without attempting to motivate the specific game, we next illustrate a graphical technique for finding (approximately) an optimum mixed strategy for any player who has only two pure strategies from which to choose. His opponent may have any (finite) number of pure strategies. Careful study of this material will facilitate understanding of §III. The example here is from Singleton and Tyndall (1974).

Consider the game with payoff matrix

	Player II Strategies				
	b_1	b_2	b_3	b_4	Row minima
a_1	1	2	4	0	0
a_2	0	-2	-3	4	-3
Column maxima	1	2	4	4	

Since the payoff matrix has no saddle-point, the problem for Player I is to find a mixture of his two pure strategies, a_1 and a_2, that maximizes his minimum expected gain (against any pure or mixed strategy of Player II). Thus, Player I seeks a probability, p, with which he will choose his strategy a_1 (and consequently a probability, 1-p, with which he will choose his strategy a_2) to maximize his minimum expected gain.

Consider that, given p, Player I's expected gain against b_1 (cf. column one of the payoff matrix) is

$$\xi_1(p) = 1p + 0(1 - p).$$

This function is plotted and labeled in the following graph. It shows, for example, that if Player I chooses p = 0, so that he chooses with certainty his strategy a_2, then he gains against b_1 exactly zero (cf. payoff matrix). If Player I chooses p = 1 so that he chooses with certainty his strategy a_1, then he gains against b_1 one. If Player I chooses an intermediate p so that he chooses a mixture of a_1 and a_2, then his expected gain against b_1 is given by the graph of $\xi_1(p)$. Thus, if Player I flips a fair coin to choose between a_1 and a_2, his expected gain against b_1 is one-half.

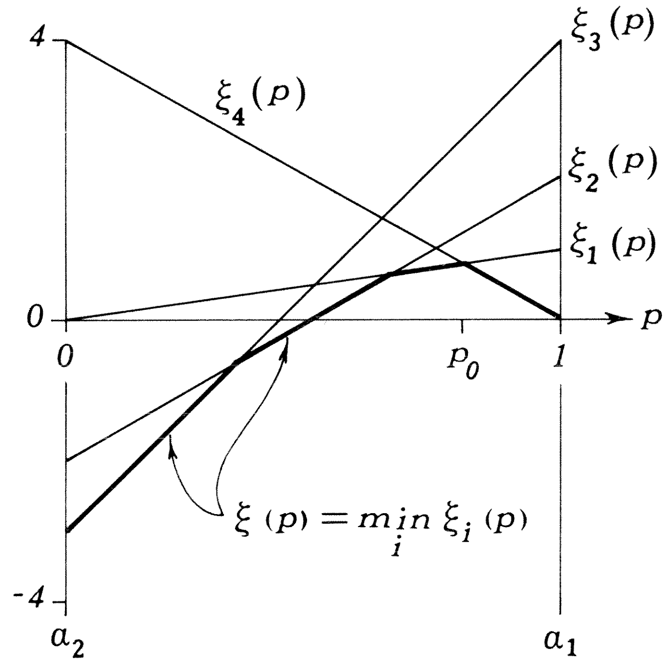
Similarly, given p, Player I's expected gain against b_2, b_3, and b_4 (cf. columns 2, 3, and 4 of the payoff matrix) are, respectively,

$$\xi_2(p) = 2p - 2(1 - p),$$

$$\xi_3(p) = 4p - 3(1 - p), \text{ and}$$

$$\xi_4(p) = 0p + 4(1 - p).$$

All of these functions are depicted in the following graph.



Points on these straight lines depict the expected gain of Player I against each of the pure strategies of Player II for any p in the interval zero to one, inclusive, i.e., for any mixed strategy for Player I.

But, the graph instantly shows more than that. For every p, it shows Player I's minimum expected gain, whatever Player II's strategy. Player I's minimum expected gain is

$$\xi(p) = \min_i \xi_i(p),$$

the bold, segmented linear function in the graph. Clearly, Player I maximizes his minimum expected gain by choosing p to maximize $\xi(p)$, i.e., by choosing the point, p_0, ticked on the graph's axis at $p = 0.8$. Thus, Player I's optimum strategy is to choose

a_1 with probability 0.8, and

a_2 with probability 0.2.

Any alternative choice of p clearly lowers Player I's minimum expected gain (cf. graph).

III. Programing for Computer Execution

Following the strategy implicit in **§II.3**, we can outline a linear programing method for the solution of general mxn games.

Consider the game with payoffs

$$\ell_{ij};\ i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n$$

where m and n are (in principle) any finite positive integers. Consider first the problem of finding an optimum strategy for Player I, i.e., an optimum vector, $\underline{p}$, of probabilities for choosing among his m pure strategies $a_1, a_2, \cdots, a_m$. Should Player I choose the particular mixed strategy (cf. **§I.5**)

$$\underline{p} = [p_1, p_2, \cdots, p_m],$$

then his expected gain against each of Player II's strategies may be written down. These expectations are:

(against b_1)

$$\xi_1(\underline{p}) = p_1 \ell_{11} + p_2 \ell_{21} + \cdots + p_m \ell_{m1};$$

(against b_2)

$$\xi_2(\underline{p}) = p_1 \ell_{12} + p_2 \ell_{22} + \cdots + p_m \ell_{m2};$$

$$\cdots$$

and

(against b_n)

$$\xi_n(\underline{p}) = p_1 \ell_{1n} + p_2 \ell_{2n} + \cdots + p_m \ell_{mn}.$$

Now, for the moment, simply define a quantity V as the minimum of these expectations, i.e., define

$$V = \min_i \{\xi_1(\underline{p}), \xi_2(\underline{p}), \cdots, \xi_n(\underline{p})\}.$$

Then (cf. **§II.3**) Player I's problem is to choose $\underline{p}$ to maximize V.

In analogy with problems of linear programing, think of Player I as having

(i) choice variables $p_1, p_2, \cdots, p_m$, and V, and

(ii) the objective of maximizing V.

Remembering the constraints on the p_i and the definition (immediately above) of V, Player I's problem is to choose $p_1, p_2, \cdots, p_m$ to:

maximize V such that

$$p_1 \ell_{11} + p_2 \ell_{21} + \cdots + p_m \ell_{m1} \geq V$$

$$p_1 \ell_{12} + p_2 \ell_{22} + \cdots + p_m \ell_{m2} \geq V$$

$$\cdots$$

$$p_1 \ell_{1n} + p_2 \ell_{2n} + \cdots + p_m \ell_{mn} \geq V$$

$$p_1 + p_2 + \cdots + p_m = 1$$

$$p_1 \geq 0$$

$$p_2 \geq 0$$

$$\cdots$$

$$p_m \geq 0$$

Now, adding a constant to all elements of the payoff matrix changes nothing as far as strategies for playing a game are concerned. In particular, the constant to be added may be chosen to be $-\max_i \min_j \ell_{ij}$. This clearly assures that the value of the game is nonnegative. Consequently, we may assume that $V \geq 0$. Then Player I's problem of finding an optimum strategy is the following linear programing program in more conventional notation:

Choose $p_1, p_2, \cdots, p_m$, V to maximize V subject to

$$
\begin{array}{rcl}
V & \geq & 0 \\
p_1 & \geq & 0 \\
p_2 & \geq & 0 \\
\cdots & & \\
p_m & \geq & 0 \\
p_1 + p_2 + \cdots + p_m & \geq & 1 \\
\ell_{11} p_1 + \ell_{21} p_2 + \cdots + \ell_{m1} p_m - V & \geq & 0 \\
\ell_{12} p_1 + \ell_{22} p_2 + \cdots + \ell_{m2} p_m - V & \geq & 0 \\
\cdots & & \\
\ell_{1n} p_1 + \ell_{2n} p_2 + \cdots + \ell_{mn} p_m - V & \geq & 0
\end{array}
$$

Player II's problem of finding an optimum strategy is the famous duality problem of linear programing.

IV. Suggested Reading

IV.1. Elementary Texts

Jones, A. J. Game theory: mathematical models of conflict. New York: Halsted Press; 1980. 309 p.

McKinsey, J. C. C. Introduction to the theory of games. New York: McGraw-Hill; 1952. 371 p.

Singleton, R. R.; Tyndall, W. F. Games and programs: mathematics for modeling. San Francisco: W. H. Freeman; 1974. 304 p.

Williams, J. D. The compleat strategyst. New York: McGraw-Hill; 1958. 234 p.

IV.2. Other Texts

Dorfman, R.; Samuelson, P. A.; Solow, R. M. Linear programming and economic analysis. New York: McGraw-Hill; 1958. 527 p.

von Neumann, J.; Morgenstern, O. Theory of games and economic behavior. Princeton, NJ: Princeton University Press; 1944. 625 p.

IV.3. Other References

Hardin, G. The tragedy of the commons. Science 162(3859): 1243-1248; 1968.

Hotelling, H. Stability in competition. In: Stigler, G. J.; Boulding, K. E., eds. Readings in price theory, Chicago: Richard D. Irwin; 1952: 467-484.

IV.4. Decision Theory in Forestry

Dane, C. W. Statistical decision theory and its application to forest engineering. Journal of Forestry 63(4): 276-279; 1965.

Davis, J. B. Forest fire control decision making under conditions of uncertainty. Journal of Forestry 66: 626-631; 1968.

Swindel, B. F. The Bayesian controversy. Res. Pap. SE-95. Asheville, NC: U.S. Department of Agriculture, Forest Service, Southeastern Forest Experiment Station; 1972. 12 p.

Thompson, E. F. The theory of decision under uncertainty and possible applications in forest management. Forest Science 14: 156-163; 1968.

Swindel, Benee F.

Theory of games and applications in forestry. Gen. Tech. Rep. SE-26. Asheville, NC: U.S. Department of Agriculture, Forest Service, Southeastern Forest Experiment Station; 1984. 13 p.

Theory has been developed covering two-person, constant-sum games of strategy. This theory is summarized and some possible applications in forestry are suggested.

KEYWORDS: Management strategy, zero-sum games, decision theory.

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