Generalization of Faustmann's Formula for Stochastic Forest Growth and Prices with Markov Decision Process Models

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ABSTRACT. Faustmann's formula gives the land value, or the forest value of land with trees, under deterministic assumptions regarding future stand growth and prices, over an infinite horizon. Markov decision process (MDP) models generalize Faustmann's approach by recognizing that future stand states and prices are known only as probabilistic distributions. The objective function is then the expected discounted value of returns, over an infinite horizon. It gives the land or the forest value in a stochastic environment. In MDP models, the laws of motion between stand-price states are Markov chains. Faustmann's formula is a special case where the probability of movement from one state to another is equal to unity. MDP models apply whether the stand state is bare land, or any state with trees, be it even- or uneven-aged. Decisions change the transition probabilities between stand states through silvicultural interventions. Decisions that maximize land or forest value depend only on the stand-price state, independently of how it was reached. Furthermore, to each stand-price state corresponds one single best decision. The solution of the MDP gives simultaneously the best decision for each state, and the forest value (land plus trees), given the stand state and following the best policy. Numerical solutions use either successive approximation, or linear programming. Examples with deterministic and stochastic cases show in particular the convergence of the MDP model to Faustmann's formula when the future is assumed known with certainty. In this deterministic environment, Faustmann's rule is independent of the distribution of stands in the forest. For. Sci. 47(4):466–474.

Key Words: Economics, risk, Markov chain, optimization, decision-making.

The theme of this article is the connection between Faustmann's (1849) formula for land valuation in forestry and modern attempts to incorporate biological and economic risk into the same problem. The intent is to show first that Faustmann's formula is a special case of a Markov decision process (MDP) model, in which the transition probabilities are unity or zero. Second, MDP theory implies independence of the decision policy from the probability distribution of states. Thus, as a deterministic limit of an MDP model, Faustmann's rule is valid regardless of the forest condition, although the effects of its application depend much on this condition.

Faustmann's model is one of the foundations of forest economics (Lofgren 1990, Samuelson 1976). It is used in numerous and varied applications, yet its validity is still questioned, even without consideration of risk. Some of its general assumptions have been criticized (Mitra and Wan 1985, 1986). And, among forest economists, there is still some debate as to whether Faustmann's formula is valid for special cases, such as a regulated forest (Oderwald and Duerr

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1990, Howard 1990, Chang 1990, Hulkrantz 1991). Several of these issues, apart from the effect of ancillary constraints which we do not consider here, are connected to the question of the dependency of Faustmann's and related rules on the initial conditions of the stand or forest under consideration. Several studies have proposed methods to solve the problem of managing a forest or a stand of trees economically given a specific initial condition. Examples for the even-aged forest include Nautiyal and Pearse (1967), and for uneven-aged forests Haight et al. (1985). The MDP framework helps clarify that although the magnitude and timing of the harvests does depend directly on the initial condition, there exists an optimal harvesting policy that ties the harvest uniquely to the current state, regardless of initial condition.

Furthermore, a serious limitation of Faustmann's traditional model is its deterministic nature. The consideration of stochastic factors is central to modern forest ecology and management (Perry and Maghembe 1989), and it is also of obvious importance in economics. Indeed, much work has been done to bring stochastic events to bear on harvesting decisions. For the case of the even-aged forest, Miller and Voltaire (1980, 1983), Clarke and Reed (1989), and Lohmander (1988) have proposed optimal stopping rules for models with risky forest growth, prices, or both. Brazee and Mendelsohn (1988), Forboseh et al. (1995), and Thomson (1992) gave special attention to the asset sale model, and the attendant calculation of a reservation price. Stochastic simulation has been used by Taylor and Fortson (1991), and by Kuboyama et al. (1997). For uneven-aged management, Haight (1990) developed a special numerical method when prices are stochastic, while Gove and Fairweather (1992) randomized the parameters of a deterministic programming model. Several other studies have dealt with risk and uncertainty, especially biological risk, in forest optimization models (Hof et al. 1996, Pickens et al. 1991).

Here, we concentrate on the Markov decision process (MDP) model as a generalization of the classic Faustmann formula. The reason for choosing MDP is its standard form, its ability to represent a wide range of stochastic processes, from catastrophic events to correlated prices, and its rich theory. In addition, several standard numerical solutions are available. Forestry applications of MDP models seem to have been first suggested by Hool (1966), but one of the first practical applications was Lembersky and Johnson's (1975) study of the optimum management of Douglas-fir plantations with risky growth and prices. Kao (1982) and Teeter and Caulfield (1991) then made similar applications. MDP models are readily adaptable to uneven-aged management, as shown by Kaya and Buongiorno (1987), and they have been also applied to forest management decisions with a mix of economic and ecological criteria, including stand and landscape diversity (Lin and Buongiorno 1998).

The article is organized as follows. First, we review briefly the simplest form of Faustmann's model, with an example to serve as a standard against which to compare the MDP formulation and results. Then, we lay out the corresponding MDP equations. We present the numerical solution of the MDP by successive approximation to illustrate (1) the general differences between MDP and Faustmann solutions, and (2) their convergence when state changes are known with certainty. We then use a linear programming approach that gives the expected discounted returns as a function of the initial probability distribution of stand states. In the deterministic case, this solution is identical to applying Faustmann's formula to each stand independently, regardless of the composition of the entire forest.

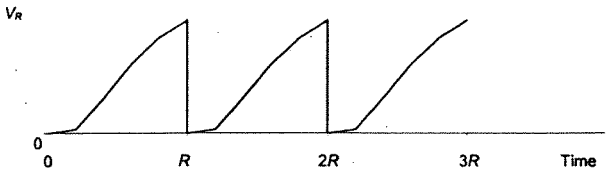

Figure 1. Stand growth and harvest in simple, deterministic Faustmann model.

The Deterministic Faustmann Formula

The classical Faustmann formula (Faustmann 1849) gives the value of a unit of bare land used in forest production, over an infinite time horizon. It is purely deterministic in terms of both its biological and economic parameters. The rate of growth of the stand of trees is assumed to be known exactly, as are all the prices and costs. It is also a static model in that the sequence and timing of decisions is always the same.

In its simplest form, the Faustmann model can be symbolized as in Figure 1. We begin at time 0, with a piece of bare land, and we establish a plantation and let it grow for R years, the rotation. All the trees are harvested at rotation age. Immediately after the harvest, a new plantation is established, identical to that of time 0. The trees then grow exactly as in the first rotation, and they are harvested at the same rotation age. This sequence is assumed to continue indefinitely.

In this simple model, let V_R be the volume of timber per unit area at age R, c the reforestation cost per unit area, p the price of timber per unit of volume net of harvesting cost, and g the discount rate per year. The function V and the parameters R, c, p, and g are assumed to be known exactly, and constant over time.

Then, the discounted value of the net income generated by an infinite series of rotations is equal to[1]:

$$SEV = \frac{pV_R - c}{(1+g)^R - 1} - c \qquad (1)$$

where the first term on the right of the equality is the sum of the infinite series of harvest income minus reforestation cost, recurring every R years, and the second term is the cost of the

[1] Faustmann's formula is sometimes written as (Clark 1990, p. 270):

$$SEV = \frac{pV_R - c}{(1+g)^R - 1}$$

but in that form, it truly expresses the value of land that has just been reforested, not the value of bare land. Still, as long as c is independent of R, optimization of this form for R does give the best economic rotation, in the sense of maximizing the present value of the return to the land.

Table 1. Faustmann's formula and economic rotation.

Rotation R (yr)	Yield V_R (m³ ha⁻¹)	Gross return $p\,V_R$	Net return $p\,V_R - c$	Discount factor $\frac{1}{(1+g)^R - 1}$	Soil expectation value $\frac{pV_R - c}{(1+g)^R - 1} - c$ (\$ ha⁻¹)
		(\$ ha⁻¹)			
20	29	377	−117	1.57	−677
40	274	3,562	3,068	0.59	1,327
60*	530	6,890	6,396	0.29	1,387
80	728	9,464	8,970	0.16	951
100	868	11,284	10,790	0.09	504

* Best rotation, R^*.
NOTE: $p = \$13\ \mathrm{m}^{-3}$, $c = \$494\ \mathrm{ha}^{-1}$, $g = 2.5\%\ \mathrm{yr}^{-1}$.

initial plantation establishment, at time 0. The result, *SEV*, or soil expectation value, is the value of bare land used in this kind of forestry. This is, in a very simplified form, the great finding of Faustmann (1849), a fundamental contribution to forestry economics and also a precursory insight into the general theory of investments (Samuelson 1976). Faustmann (1849) also gave an extension of formula (1) to calculate the value of a stand with trees younger than the best rotation.

Equation (1) has numerous applications, to compare different land uses, or to compare different forest management strategies. For that purpose, it usually includes much more detail such as density in plantations, commercial and precommercial thinning, and so on. As an example of application, Equation (1) can be used to find the best economic rotation, R, i.e., the rotation that maximizes the land value, *SEV*. If V_R is a simple continuous function, the best rotation, R^*, can be found with the calculus. But numerical solutions are used usually in more complex cases, along the lines illustrated in Table 1, where each row shows the *SEV* for a particular R. With the data used in the table[2] the best rotation, in the sense of maximizing soil expectation value, obtains around the age of 60 yr.

Markov Decision Process Model

Equation (3) is obviously a very crude representation of reality, a dreadful world in which the future is like the past, forever. It is well known, instead, that biological systems, including forests, are shaped in large part by rare catastrophic events, and by milder but more frequent shocks. These are either truly random events, or they must be assumed so because of lack of knowledge. The same is even truer of economic variables. Prices of forest products and cost of production may vary widely over time, so that they also are best modeled as stochastic processes.

The stochastic formulation of Faustmann's formula is the analog of the deterministic formulation (1). That is to say, it seeks the sequence of decisions that maximizes the expected discounted value of the income, over an infinite horizon, given a specific initial condition. So, the general formulation of the objective function is:

$$\max_{d_t} V(X_0) = E[\sum_{t=0}^{\infty} r(X_t, d_t)\beta^t \mid X_0] \tag{2}$$

where X_t designates the system state at t, which instead of being deterministic is now a random variable; d_t is the decision at time $t = 1, 2, \ldots$; r is the immediate return from the decision given a particular state; and β is the discount factor $\beta = (1+g)^{-T}$, where g is the interest rate, and T is the number of years between decisions.

There are many different formulations of this problem, with particular solutions according to the specification of the law of motion that ties X_t to X_{t-1}, d_{t-1}, and of the random shocks. However, a very general approach, most practical, and with a vast array of theoretical and numerical results, is the Markov decision process (MDP) model (Hillier and Lieberman, 1990, p. 768–802). Applied to Faustmann's problem, assuming only stochastic growth for simplicity, the model replaces the deterministic growth function V_R by a stochastic growth process described by a matrix of transition probabilities. Each matrix element is the probability that the system moves from one state to another between t and $t+1$. The state domain consists of N discrete states.

More generally, state definitions may involve biological as well as economic variables. For example, Lembersky and Johnson (1975) used number of trees and average diameter to define the state of even-aged stands, and four levels of autocorrelated prices for market states. Lin and Buongiorno (1998) used basal area, tree size, and species to characterize the states of uneven-aged stands, and two levels of white-noise prices.

The transition probabilities between system states are often computed by stochastic simulation, based on growth and econometric models (Kaya and Buongiorno 1987). A decision is an action (e.g., tree removal or planting) that causes the system to move from a state to another, thus changing the transition probabilities between states.

Transition probabilities are stationary and depend only on the current system state. As shown by Taylor (1984), the Markov chain is a very general model of price changes.[3]

[2] The yield data in Table 1 are for Douglas fir in the Pacific Northwest, site I (McArdle et al. 1961, p. 24).

[3] "A Markovian price expectation structure refers to any stochastic model in which price is conditional on previous prices; hence, the assumption embraces random walk, rational expectations, autoregressive, and many other conditional models" (Taylor 1984).

Table 2. Definition of stand states.

State i	Volume (m^3 ha^{-1})
0*	0
1	29
2	274
3	530
4	728
5	868

* Bare land.

Concerning biological growth, some studies have argued that forest growth is not stationary and Markovian (Binkley 1980), but this depends largely on the finesse of the state definitions. Higher than first order Markovian relations can be incorporated to model biological or price changes by adding appropriately defined state variables (e.g., making the last change part of the state definition). If stand and price states are independent, which is often the case though it is not required by the Markov model, it is a simple matter, as shown in footnote 4, to obtain from the separate transition probabilities of stands and prices the transition probabilities between combined stand-price states.

To illustrate the formulation of the MDP model, consider the case of a plantation similar to the one used to illustrate Faustmann's deterministic formula (Table 1). For simplicity, and without loss of generality, stand growth is the only stochastic element. The volume per hectare (Table 2) defines the stand state. To facilitate comparison with the deterministic Faustmann formula, the volume per area in state 1 is the same as V_{20} in Table 1, that in state 2 is the same as V_{40}, and so on. Furthermore, we assume that the interval between decisions, t to $t + 1$, is 20 yr. In parallel with Faustmann's model, the decision is to do nothing, or to cut the stand and reforest immediately.

The second element of the MDP model, the reward structure, is described in Table 3. If the decision is to do nothing, the reward is zero, regardless of stand state. If instead the decision is to harvest and reforest immediately, the reward is an immediate return equal to the net value of the timber harvested, minus the reforestation cost. Again, as in the deterministic Faustmann formula, the price of timber is \$13 m^{-3}, and the reforestation cost is \$494 ha^{-1}. The real interest rate is 2.5% per year. So, the discount factor is $\beta = (1 + g)^{-20} = 0.61$.

Concerning stand growth, instead of it being certain and dictated by the deterministic function V_R once the stand is established, the periodic growth of the stand is stochastic, defined by probabilities of passage from one state to another. To continue the example, Table 4 shows that starting with bare land (state 0) and doing nothing obtains bare land in 20 yr with probability 1. If instead the stand is in state 1 (29 m^3 ha^{-1}), there is a 70% probability of finding it in state 2 (274 m^3 ha^{-1}) in 20 yr, a 10% probability of finding it in state 1, or 3, and also a 10% probability of finding it back to state 0, due to a major catastrophe. In this example, the probability of a catastrophic event is constant over time, about 0.5% in any year, and independent of the stand state.

Table 3. Immediate return as a result of state and decision $\{r_{ik}\}$.

State i	Decision k* 0	c
	(\$ ha^{-1})	
0	0	−494
1	0	−117
2	0	3,068
3	0	6,396
4	0	8,970
5	0	10,790

* 0 = do nothing, c = cut and reforest.

Table 5 shows instead the probability of transition between states if the decision is to cut the stand and to reforest immediately. Regardless of the initial stand state, the probability of ending in state 1 after 20 yr is 90%, while the possibility of catastrophic events translates in a 10% probability of reverting to state 0. A decision, then, consists in choosing the transition probabilities in Table 4, or those in Table 5, with the attendant immediate reward and future consequences. A policy is a rule specifying how each period decision is taken. A fundamental result is that an optimal policy exists and that there is always a stationary policy that is optimal (Blackwell 1962). In a stationary policy, the decision depends only on the system state and is independent of time.

Solution by Successive Approximation

The optimal policy can be obtained in different ways, including the policy improvement algorithm, linear programming, and successive approximation (Hillier and Lieberman 1990, p. 784–792). Successive approximation is intuitively appealing, because it is based on the classical principle of optimality of dynamic programming. It is also numerically efficient because it does not require the solution of simultaneous equations. Letting V_i^t be the maximum expected long-run discounted returns for a stand now in state i and that will grow for t periods, V_i^t satisfies the backward recursive equations:

$$V_i^t = \max_k [r_{ik} + \beta \sum_{j=1}^{N} p(j \mid i,k) V_j^{t-1}] \qquad i = 1,\ldots,N$$
$$V_i^0 = 0 \tag{3}$$

Table 4. Probability of transition between stand states, doing nothing $\{p(j|i,0)\}$.

Begin state i	End state j 0	1	2	3	4	5
0	1.00					
1	0.10	0.10	0.70	0.10		
2	0.10		0.10	0.70	0.10	
3	0.10			0.10	0.70	0.10
4	0.10				0.10	0.80
5	0.10				0.10	0.80

Table 5. Probability of transition between stand states, with cut followed by reforestation $\{p(j|i,c)\}$.

Begin state i	End state j 0	1	2	3	4	5
0	0.10	0.90				
1	0.10	0.90				
2	0.10	0.90				
3	0.10	0.90				
4	0.10	0.90				
5	0.10	0.90				

Table 6. Successive approximation of maximum expected return and best decision, stochastic case.

	Maximum expected return ($\$ha^{-1}$)						Best decision*					
Iteration t	V^t_0	V^t_1	V^t_2	V^t_3	V^t_4	V^t_5	d_0	d_1	d_2	d_3	d_4	d_5
0	0	0	0	0	0	0						
1	0	0	3,068	6,396	8,970	10,790	0	0	c	c	c	c
2	0	1,701	3,467	6,396	8,970	10,790	0	0	0	c	c	c
3	440	1,975	4,002	7,330	9,904	11,724	c	0	c	c	c	c
4	618	2,304	4,180	7,508	10,082	11,902	c	0	c	c	c	c
5	809	2,422	4,371	7,699	10,273	12,093	c	0	c	c	c	c
6	886	2,534	4,448	7,776	10,350	12,170	c	0	c	c	c	c
7	952	2,583	4,514	7,842	10,416	12,236	c	0	c	c	c	c
8	983	2,623	4,545	7,873	10,447	12,267	c	0	c	c	c	c
9	1,006	2,642	4,568	7,896	10,470	12,290	c	0	c	c	c	c
10	1,019	2,656	4,581	7,909	10,483	12,303	c	0	c	c	c	c
...	...	...	...	...	...	...	...	...	...	...	...	...
19	1,039	2,676	4,601	7,929	10,503	12,323	c	0	c	c	c	c
20	1,039	2,676	4,601	7,929	10,503	12,323	c	0	c	c	c	c

* $d_i = 0$ means do nothing in state i, d_i = c means cut and reforest.

where $p(j \mid i,k)$ is the probability that the stand[4] will be in state j at $t + 1$, given state i at t, followed by decision k. r_{ik} is the immediate return from decision k for a stand in state i, and β is the discount factor, defined above.

The equations mean that during the current period, with t periods to go, we earn the immediate return r_{ik}, plus the discounted value of the future returns that we will earn from the $t - 1$ subsequent periods by following the optimal policy. The initial condition V_i^0 is arbitrary and usually set to zero for each state i. As t increases to infinity, V_i^t converges to V_i, the expected discounted return from a stand that starts in state i, and that is treated according to the best policy. The rate of convergence is rapid, and can be improved further by the error bound method (Bertsekas 1995, vol I, p. 308–309). In particular, V_0^t converges towards V_0, the maximum expected discounted return from bare land. V_0 is then the soil expectation value with stochastic growth (and with stochastic prices if the model had several price states). Thus, Equation (3) can be viewed as a generalization of Faustmann's formula for a stochastic environment. Faustmann's formula is a special case in which a few of the transition probabilities are equal to 1 and all others to zero, reflecting the fact that in a deterministic case we move from one state to another with certainty. For stand state with trees, the solution gives the forest value, inclusive of land and trees, in analogy with Faustmann's extension of his formula to immature stands (Faustmann 1849).

As Faustmann's formula gives the best rotation in addition to the land value, the MDP model also gives the best policy, i.e., the best decision for each state. The best policy is stationary. This property helps clarify the issue of how to economically cut a stand or forest in a particular state. It is a fundamental result of the MDP analysis that the optimum policy is invariant over time and is tied only to the current system state. Since, as we have seen, Faustmann's formula is a special case of the MDP model, it follows that the best rotation in the deterministic case is valid regardless of stand state.[5]

To pursue the example, Table 6 shows several iterations in the solution of the recursive Equation (4), to find the best management policy (decision for each state), and the corresponding maximum expected discounted returns, over an infinite horizon, given a particular initial state. The V_i^0 condition was set arbitrarily at zero, for all stand states. The computations were done with a spreadsheet. Convergence occurred quickly. After 20 iterations, the V_i^t's were stable. The best policy was already obtained after three iterations: reforest in state 0, do nothing in state 1, harvest and reforest immediately in states 2, 3, 4, 5. This would correspond to a rotation of 40 yr, 20 yr shorter than the one obtained with Faustmann's formula. The maximum expected discounted return starting from bare land and following the best policy was about $1,039, also lower than the best soil expectation value obtained with the Faustmann formula in Table 1, due to the presence of risk.[6] A feature of the successive approximation solution is that it gives simultaneously the forest values, land plus trees, for states 1 to 5 as well.

To illustrate how the Markov model gives the Faustmann solution as a special case, the transition probabilities were changed as in Tables 7 and 8. The data in Table 7 mean that if nothing is done, a stand in state 0 is still in state 0 after 20 yr. A stand in state 1, moves to state 2 with certainty; if in state 2, then it moves to state 3 with certainty; and so on. Table 8 says that a stand in state 0 (bare land) that is reforested, or a stand in any other state that is cut and reforested, moves always to state 1 in 20 yr.

[4] If the system state $i = (s,m)$ consists of the stand state s and the price state m, and $j = (s',m')$ is another state, then the probability of moving from state i to j with decision k is: $p(j|i,k) = p(j|s,k)p(m'|m)$, if stand and price are independent. The price can be random noise, in which case $p(m'|m)$ depends on m' only (e.g., Kaya and Buongiorno 1987), or it can be autoregressive, in which case $p(m'|m)$ depends on m' and m (Lembersky and Johnson 1975). But in fact, the Markov model is more general (Taylor 1984). The Markov model can also account for correlation between price and market state. In all cases, the fundamental model (3) remains the same, only the definition of the state space changes. In our example it is assumed, for simplicity but without loss of generality, that the stand growth only is stochastic.

[5] This would not generally be true with additional constraints, such as a requirement that production be constant over time.

[6] However, the MDP solution would always give a land value at least equal to Faustmann's formula, if the latter were solved with returns equal to the expected immediate returns implied by the stochastic formulation (Norstrom 1975).

Table 7. Transition probability between stand states in deterministic case, doing nothing $\{p(j|i,0)\}$.

Begin state i	End state j: 0	1	2	3	4	5
0	1.00					
1			1.00			
2				1.00		
3					1.00	
4						1.00
5						1.00

Table 8. Transition probability between stand states in deterministic case, with cut followed by reforestation $\{p(j|i,c)\}$.

Begin state i	End state j: 0	1	2	3	4	5
0		1.00				
1		1.00				
2		1.00				
3		1.00				
4		1.00				
5		1.00				

Therefore, the transition probabilities in Tables 7 and 8 are equivalent to the deterministic growth function in Table 1. Furthermore, the objective function is the same: maximization of discounted returns over an infinite horizon. With the same reward structure, the Faustmann formula and the deterministic Markov model should give the same results. Table 9 shows the solution of the Markov deterministic model, by successive approximation. After 20 iterations, the maximum discounted value of returns over an infinite horizon stabilizes. The best policy is to reforest in state 0, do nothing in state 1 and 2, and cut and reforest in states 3, 4, and 5. With our definition of stand states, this is equivalent to a 60 yr rotation, equal to what was found with Faustmann's formula (Table 1). With this policy, the maximum discounted value of returns from state 0 is \$1,387 ha^{-1}, also equal to the soil expectation value given by Faustmann's formula, as it should. In addition to the soil expectation value, the Markov deterministic approach also shows the forest value for each possible initial state. The highest forest value comes naturally for a stand with the largest initial growing stock (State 5).

Solution by Linear Programming

Although the linear programming solution of the Markov decision problem is less straightforward than the successive approximation approach, which uses standard dynamic programming concepts, it is worth study. It is numerically compact, has a more precise solution than successive approximation, and more importantly, it gives additional insight. There are at least two linear programming approaches, d'Epenoux (1960) and Ross (1983 p. 40–42). We present d'Epenoux's because it is numerically more efficient, with only as many constraints as there are states, while Ross's has $N \times K$ constraints, where N is the number of states and K the number of decisions. Furthermore, the objective function in d'Epenoux's model is the expected discounted return per unit area for a given initial distribution of stand states. Thus, it gives directly the value of a forest consisting of stands in many different states, managed optimally.

D'Epenoux linear programming formulation of the Markov decision process model consists in finding:

$$\max_{y_{ik}} FV = \sum_{i=1}^{N}\sum_{k=1}^{K} r_{ik} y_{ik}$$

$$\sum_{k=1}^{K} y_{jk} - \beta \sum_{i=1}^{N}\sum_{k=1}^{K} y_{ik} p(j \mid i,k) = \pi_j \qquad j = 1,\ldots,N \tag{4}$$

$$y_{ik} \geq 0$$

where the variable y_{ik} can be interpreted as a "weighted (in a discounted sense) expected time of being in state i and making decision k" (Hillier and Lieberman 1990, p. 787). β is the discount factor defined above, and π_j is the probability of the initial state j. Thus, the objective function is the expected long-term discounted return, i.e., the forest value, FV, given the probability distribution of the initial stand state.

Table 9. Successive approximation of maximum expected return and best decision, deterministic case.

Iteration t	Maximum expected return (\$ha^{-1}): V_0	V_1	V_2	V_3	V_4	V_5	Best decision*: d_0	d_1	d_2	d_3	d_4	d_5
0	0	0	0	0	0	0						
1	0	0	3,068	6,396	8,970	10,790	0	c	c	c	c	c
2	0	1,872	3,903	6,396	8,970	10,790	0	0	0	c	c	c
3	649	2,382	4,211	7,539	10,113	11,933	c	0	c	c	c	c
4	960	2,570	4,601	7,850	10,424	12,244	c	0	0	c	c	c
5	1,074	2,808	4,790	7,964	10,538	12,358	c	0	0	c	c	c
6	1,219	2,923	4,860	8,109	10,683	12,503	c	0	0	c	c	c
7	1,290	2,966	4,949	8,180	10,754	12,574	c	0	0	c	c	c
8	1,316	3,020	4,992	8,206	10,780	12,600	c	0	0	c	c	c
9	1,349	3,047	5,008	8,239	10,813	12,633	c	0	0	c	c	c
10	1,365	3,056	5,028	8,255	10,829	12,649	c	0	0	c	c	c
...	...	...	...	...	...	...	...	...	...	...	...	...
19	1,387	3,082	5,051	8,277	10,851	12,671	c	0	0	c	c	c
20	1,387	3,083	5,051	8,277	10,851	12,671	c	0	0	c	c	c

* $d_i = 0$ means do nothing in state i, $d_i = c$ means cut and reforest.

Table 10. Linear programming solution of forest value and best decision, stochastic case.

Variable:	y_{00}	y_{10}	y_{20}	y_{30}	y_{40}	y_{50}	y_{0c}	y_{1c}	y_{2c}	y_{3c}	y_{4c}	y_{5c}	
Solution:		1.06					0.32		0.62	0.23	0.17	0.17	
Objective: (max)							–494	–117	3,068	6,396	8,970	10,790	= 6,512 ($ ha^{-1})
Constraints:													
State 0	0.39	–0.06	–0.06	–0.06	–0.06	–0.06	0.94	–0.06	–0.06	–0.06	–0.06	–0.06	= 1/6 = π_0
State 1		0.94					–0.55	0.45	–0.55	–0.55	–0.55	–0.55	= 1/6 = π_1
State 2		–0.43	0.94						1.00				= 1/6 = π_2
State 3		–0.06	–0.43	0.94						1.00			= 1/6 = π_3
State 4			–0.06	–0.43	0.94	–0.06					1.00		= 1/6 = π_4
State 5				–0.06	–0.49	0.51						1.00	= 1/6 = π_5
Decision*:	d_{00}	d_{10}	d_{20}	d_{30}	d_{40}	d_{50}	d_{0c}	d_{1c}	d_{2c}	d_{3c}	d_{4c}	d_{5c}	
		1					1		1	1	1	1	

* $d_{ik} = 1$ means decision k is best in state i.

Given a solution of the linear program, the best policy is defined by the probability of making a decision in a particular state:

$$d_{ik} = \frac{y_{ik}}{\sum_{k=1}^{K} y_{ik}} \tag{5}$$

The best policy is deterministic, i.e., $d_{ik} = 0$ or 1. Furthermore, the best policy is independent of the initial probability distribution of stand states, i.e., the π_i's, although the forest value depends very much on this initial condition.

The formulation of the linear programming model for the example used so far is in Table 10, together with the best solution. The coefficients of the objective function are the rewards in Table 3. The coefficients of the constraints were calculated with the transition probabilities in Tables 4 and 5, according to Equation (4). The right-hand side of the constraints is the probability of the initial state, set arbitrarily equal to 1/6 for every state.

The second row in Table 10 shows the best solution for the y_{ik}'s. The corresponding objective function optimum, \$6512 ha^{-1}, is the expected net present value of returns over an infinite horizon, conditional on the initial probabilities. The best decisions, in the last row of Table 10, are the probabilities of each action, given a particular state, computed with Equation (5). These decisions are independent of the initial probability distribution of states. The decisions are nonrandomized (0 or 1); thus in every state the same decision is always called for.

The best policy (decision by state) in Table 10 is the same as that obtained by successive approximation (Table 6), as it should given the same data. The objective function of the linear programming solution is related to the maximum long-run discounted return given by the successive approximation approach for each initial state, since:

$$6512\ (\$\ \text{ha}^{-1}) = 1/6(1039 + 2676 + 4601 + 7929 + 10503 + 12323),$$

which verifies that the linear programming objective function gives directly the forest value per unit area, for a given distribution of initial forest states.

The linear programming solution of the deterministic Markov problem, i.e., with the transition probabilities in Tables 7 and 8, is in Table 11. It gives the same decision rule as the successive approximation approach (Table 9). The objective function, \$6,887 ha^{-1} is the maximum discounted value of returns per hectare for a forest that has 1/

Table 11. Linear programming solution of forest value and best decision, deterministic case.

Variable:	y_{00}	y_{10}	y_{20}	y_{30}	y_{40}	y_{50}	y_{0c}	y_{1c}	y_{2c}	y_{3c}	y_{4c}	y_{5c}	
Solution:		0.82	0.67				0.17			0.57	0.17	0.17	
Objective: (max)							–494	–117	3,068	6,396	8,970	10,790	= 6,887 ($ ha^{-1})
Constraints:													
State 0	0.39						1.00						= 1/6 = π_0
State 1		1.00					–0.61	0.39	–0.61	–0.61	–0.61	–0.61	= 1/6 = π_1
State 2		–0.61	1.00						1.00				= 1/6 = π_2
State 3			–0.61	1.00						1.00			= 1/6 = π_3
State 4				–0.61	1.00						1.00		= 1/6 = π_4
State 5					–0.61	0.39						1.00	= 1/6 = π_5
Decision*:	d_{00}	d_{10}	d_{20}	d_{30}	d_{40}	d_{50}	d_{0c}	d_{1c}	d_{2c}	d_{3c}	d_{4c}	d_{5c}	
		1	1				1			1	1	1	

* $d_{ik} = 1$ means decision k is best in state i.

6 of its area in each of the possible states, and which is managed perpetually according to the optimum rule. In this deterministic situation, the optimum policy is the same as Faustmann's: do nothing in states 1 and 2, reforest in state 0, cut and reforest in states 3, 4, and 5.

Indeed, the same forest value can also be obtained by applying Faustmann's best rotation to a forest with 1/6 of the area in each of the possible states, as shown in Table 12. Following Faustmann, bare land has the *SEV* given by Equation (1). Immature stands, i.e., stands of age T less than the best rotation R^*, are let grow for R^*-T years, before being replaced, so that their forest value (land plus trees) is:

$$FV = \frac{1}{(1+g)^{R^*-T}}(pV_{R^*} + SEV) \tag{6}$$

While mature or overmature stands, of age $T \geq R^*$ are harvested immediately, so that their forest value is:

$$FV = pV_T + SEV \tag{7}$$

In our example, the best rotation according to Faustmann's formula is 60 yr (Table 1). With the same data, Equations (6) and (7) give the results in Table 12. For example, a 20 yr old stand is worth \$3,083 ha^{-1} according to Equation (6), exactly the same as the forest value given by the deterministic Markov model for the initial state 1 (Table 9).

If the forest area is distributed equally so that each age-class covers 1/6 of the land, then the average value per acre of this forest is \$6,887 ha^{-1}, which is also the value of the objective function of the linear programming solution of the MDP when the initial states are evenly distributed with probability 1/6 (Table 11). This correspondence holds true regardless of the probability distribution of initial states, thus confirming the validity of Faustmann's formula independently of the forest condition, and in particular for a regulated forest structure, as previously argued by Chang (1990). That Faustmann's model is a special case of the MDP model with deterministic laws of motion gives more generality to this result. Faustmann's model is a particular type of MDP, thus Faustmann's policy must be independent of forest condition, be it a regulated forest or any other kind of stand distribution. However, this would not be generally true with additional constraints, such as constant periodic production.

Table 12. Forest value of regulated forest according to Faustmann's formula.

Stand age	Value (\$ ha^{-1})	Area
0*	1,387	1/6
20†	3,083	1/6
40†	5,051	1/6
60†	8,277	1/6
80††	10,851	1/6
100††	12,671	1/6
Average:	6,887	

* Bare land.
† Immature stands.
†† Mature stands.

Summary and Conclusion

Faustmann's formula gives the value of forest land, under deterministic assumptions regarding future stand growth and prices, over an infinite horizon. Markov decision process models generalize Faustmann's approach by assuming that future stand states and prices are known only as probabilistic distributions. The objective function is then the expected long-run discounted value of returns. This is the land value, or the forest value if there is an initial stock, in a stochastic environment. In MDP models, the laws of motion between system states are Markov chains. By proper definition of the state space, MDP models can incorporate biological as well as economic risk, in even-aged, uneven-aged or mixed systems.

The traditional deterministic Faustmann formula can be interpreted as a special case of the more general MDP model where the probability of movement from one state to another is equal to zero or unity. Furthermore, MDP models deal simultaneously with all possible initial states, be it bare land, or land and trees. Decisions consist in moving a stand from one state to another through silvicultural interventions. The theory of Markovian decision processes shows that the best decision depends only on the current stand-market state, independently of how it was reached. Furthermore, the best decision is nonrandomized: to each stand-price state corresponds one best decision with a probability of one.

The solution of the MDP gives simultaneously the best decision policy and the stand value (land plus trees), given the initial state followed by the optimum decision policy. Numerical solutions may use successive approximation, or linear programming. The results for a simple even-aged forest, with deterministic or stochastic growth have illustrated the convergence of the MDP results with those of Faustmann's formula when the current state is bare land or land with trees, and the future is known with certainty. More importantly, MDP theory demonstrates the existence of stationary optimal decision policies. Faustmann's rule is a special case, and it is therefore valid (within the real limitations of the deterministic assumption) regardless of the initial distribution of stands, regulated or not.

The advantage of MDP models is their generality and realism. Randomness, a key factor in biology and economics, is absent from Faustmann's model. The MDP discretization of the state space fits the foresters' practice of classifying stands by "type." Very fine state spaces are possible, and solvable by either successive approximation, which does not require the solution of simultaneous equations, or by linear programming which can handle a great number of constraints and variables. Constrained MDP systems are also possible, since many constraints simply mean a restriction of the state space (Lin and Buongiorno 1998). Although MDP's have been discussed here in relation to Faustmann's formula, they offer a general means of reducing much larger and complex stochastic systems to elegantly simple probability matrices readily amenable to analysis and optimization (Holling et al. 1996).

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