THE EFFECT OF DENSITY ON THE HEIGHT-DIAMETER RELATIONSHIP

Boris Zeide and Curtis Vanderschaaf[1]

Abstract—Using stand density along with mean diameter to predict average height increases the proportion of explained variance. This result, obtained from permanent plots established in a loblolly pine plantation thinned to different levels, makes sense. We know that due to competition, trees with the same diameter are taller in denser stands. Diameter and density are not only necessary, but may be sufficient for determining tree height because other factors affecting height are reflected by diameter and density. In the process of developing the proposed model we found that height increases monotonically with density and that this increase is not bounded by an asymptote. Contrary to our expectations, the inclusion of density did not bring the allometric parameter of diameter closer to the theoretical value of 2/3.

INTRODUCTION

By relating height and diameter we can express height from diameter, which can be measured easier and more reliably. This relationship also informs us about stem taper and, therefore, volume. As a result, the height-diameter relationship is one of the most studied in forestry. Although diameter is a good predictor of height, we may advance further by using other available information. Diameter explains a lot of variation in height. After all, it is designed to support the load that depends on tree height. Still, there are other factors determining the load that may modify the height-diameter relationship. The most obvious among these factors is stand density.

APPROACH AND BASIC ASSUMPTIONS

Theoretical and empirical studies of the height-diameter relationship suggest that it is an allometric function with the power of diameter, *b*, equal to 2/3 (Greenhill 1881, McMahon 1973, Norberg 1988, O'Brien and others 1995):

$$H = aD^b \qquad (1)$$

This relationship describes a column of equal resistance to bending and buckling, which is a reasonable assumption for tree stems exposed, in addition to the force of gravity, to wind (O'Brien and others 1995, Schniewind 1962) and snow (King and Loucks 1978, McMahon and Bonner 1983). Such a column maintains elastic similarity along the stem (Rich 1986, Rich and others 1986). Elastic similarity leads to b=2/3 and allows the tree to maintain a constant safety factor against both buckling and bending due to tree weight and wind force (McMahon and Bonner 1983, Norberg 1988, Rich and others 1986).

Besides purely structural considerations, there is a biological component. Trees have evolved to equalize not so much the strength along the stem as to equalize the damage to its survival. Below the crown this biological requirement coincides with the mechanical one because at any point breakage dooms the tree. The situation inside the crown is different. Trees may survive the loss of a third of the crown and more. Therefore, it would not pay to invest into equal strength of the upper stem. Indeed, trees often lose tree tops, most frequently within the upper third of the crown.

Equation (1) assumes that height depends exclusively on diameter. This is not true: in dense stands trees with the same diameter are taller than those in less dense stands. Therefore, stand density should be included as the second predictor of average height. Out of many ways to incorporate density into the predicting equation, we tested several asymptotic and non-asymptotic density modules (table 3).

As the measure of density we used Reineke's Stand Density Index (SDI) (Reineke 1933):

$$SDI = N^*(D/10)^{1.7} \qquad (2)$$

where: N = number of trees per acre, D = quadratic mean diameter of a stand. The power of 1.7 was provided by MacKinney and others (1937) who reanalyzed the data used by Reineke (1933) with standard statistical methods. Sometimes it is convenient to normalize the index by dividing it by the maximum value of 450 which was reported by Reineke for loblolly pine:

$$I = (N^*(D/10)^{1.7})/450 \qquad (3)$$

Density does not affect height prior to the onset of competition, which happens, according to our observations when Reineke's index is 34. This minimal level of density, denoted as I0 = 34/450, is used in the following models to set the initial effective density to 1.

[1]Professor and Research Specialist, School of Forest Resources, University of Arkansas, Monticello, AR 71656-3468, respectively.

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Table 1—Quadratic mean diameter (D - in inches) and average total tree height (H - in feet) by TBA (target basal area in square feet per acre) treatment from the Monticello thinning and pruning study. No measurements were conducted prior to age 27 for the Control TBA, and no height measurements were conducted at age 37 for any density. The Control TBA had an average basal area of 137 square feet per acre across all

	TBA									
Age	-- 30 --		-- 50 --		-- 70 --		-- 90 --		**Control**	
	D	H	D	H	D	H	D	H	D	H
12	6.0	36.4	6.9	37.2	6.6	36.6	6.6	35.9	.	.
15	9.1	43.6	8.7	44.2	8.0	43.7	7.8	43.3	.	.
16	9.7	46.3	9.4	48.1	8.6	46.8	8.2	46.0	.	.
19	11.7	51.8	11.1	51.5	9.9	51.2	9.4	49.0	.	.
24	14.4	61.2	13.5	64.3	12.0	62.8	11.1	61.0	.	.
27	16.2	67.2	15.0	69.4	13.4	68.7	12.3	67.1	9.9	58.2
30	18.4	73.5	16.8	75.5	14.9	74.5	13.7	74.2	10.8	65.7
35	21.1	77.5	19.1	80.3	16.9	79.8	15.4	79.7	11.6	72.8
37	21.8	-	19.7	-	17.4	-	16.0	-	12.6	-
40	22.7	84.7	20.6	86.6	18.3	86.0	16.7	86.5	12.8	78.6

DATA

We used data collected during ten measurements on 40 permanent plots (table 1) established in 1970 by the Southern Forest Experiment Station in a typical 12-year-old loblolly pine (*Pinus taeda L.*) plantation in southeast Arkansas (Burton 1981). This is the second oldest active thinning and pruning study in loblolly pine stands. What makes these data particularly suitable for this research is the wide range of density. Plots were initially thinned at age 12 to 40, 60, 80 and 100 feet^2/ac of basal area. After the second inventory at age 15, basal areas were reduced to 30, 50, 70 and 90 feet^2/ac (TBA) and maintained at these levels by subsequent thinnings at ages 24, 27, 30, 35, and 40. The density variation was further enhanced by three severe ice storms. At the age of 27 five control plots (without thinning) were established on the adjacent untreated portion of the plantation.

DOES DENSITY HELP TO PREDICT HEIGHT WHEN DIAMETER IS KNOWN?

Before designing a model to predict height using diameter and density, we would like to make sure that the density effect is significant. Two methods were used for this purpose. First, we fitted the traditional allometric model relating height and diameter (equation (1)) to five groups of plots differing in density. The equation was linearized by log-transforming the variables. We found that predicted average heights of the stands with the same diameter (average quadratic mean diameter across all treatments and ages) increased with stand density level (table 2). The height difference between the extreme levels of density is 21 feet. Parameter *b* also showed an increasing trend in managed stands. Its pooled value is 0.7374, which is slightly greater than 2/3, probably because of the unaccounted effects of density.

The second method is to test several models including density as a predictor along with diameter (table 3). In all tested models the parameter of density was significantly different from 0, which indicates that, regardless of equation form, density does help to predict height when diameter is known. The inclusion of density increased the proportion of explained variation in height from 0.88 to 0.93.

IS THERE AN OPTIMAL DENSITY FOR HEIGHT GROWTH?

Now that we are sure that density is an important predictor of height, we want to know whether there is a density at which height reaches its maximum for a given diameter. Discovering such an optimal density would be of help to foresters who are interested in maximizing height growth.

To solve this question, we used a model flexible enough to locate a possible culmination of height. To this end, our model includes two density terms, driven by density (I), and density squared (I^2):

$$H = aD^{b}e^{cI+g(I*I)} \qquad (4)$$

If *c* and *g* are both positive, there is no maximum height. If *c* and *g* are both negative, then our logic and analytical procedures are entirely incorrect because this would mean that height decreases when density increases. But, if *c* is positive and *g* is negative then there is a maximum height.

The results (*c* = 0.9412 *g* = -0.6180) show that there is an optimal density, that is the density at which height reaches a maximum. This conclusion contradicts our understanding of the involved processes. We believe that when diameters are equal, average height increases with increasing density. Should we trust the parameter values obtained from a limited data set or our reasoning? Fortunately, this contradiction can be resolved by calculating the value of the

Table 2—Comparison of the relationship between height and diameter by density treatment fitted to data from the Monticello thinning and pruning study. Where D = quadratic mean diameter in inches, H = average height in feet (height corresponding to D), Obs. = number of observations, Den = square feet of basal area per acre, SEE = standard error of the estimate, Hest = average height in feet estimation of a stand with a QMD of 13 inches (average size of D across all treatments and ages), SEE = standard error of the estimate, Adj. R^2 - is the adjusted R-squared value. Variables were log-transformed prior to fitting. The number after ± represents the single standard error

Equation	Obs.	Den	a	b	SEE	Adj. R^2	Hest
$H = aD^b$	90	32	9.7834 ±0.4197	0.6855 ±0.0160	0.0596	0.9538	57
$H = aD^b$	90	51	8.3875 ±0.3441	0.7730 ±0.0157	0.0533	0.9646	61
$H = aD^b$	90	69	7.8677 ±0.2680	0.8259 ±0.0136	0.0436	0.9764	65
$H = aD^b$	90	85	6.5471 ±0.2638	0.9194 ±0.0165	0.0486	0.9720	69
$H = aD^b$	19	137	9.3161 ±2.8621	0.8263 ±0.1105	0.0598	0.7532	78

optimal density, I^*, which can be obtained from the following equation:

$$dH/dI = H(c+2gI) = 0 \quad (5)$$

Hence $I^* = -c/2g = 0.7615$. This value is beyond the data range: the actual maximum density of the data is 0.7017. This means that the discovered optimum is illusory. The negative term indicates that the relationship between height and density is not linear but concave down.

IS THE RELATIONSHIP BETWEEN DENSITY AND HEIGHT ASYMPTOTIC?

The next question is: does the discovered concave form approach a finite maximum height or is the height increase unlimited? The asymptotic form means that when density is high further increase will produce practically no increase in height, which is not likely. We believe that the non-asymptotic form is more biologically reasonable. Besides this somewhat intuitive reasoning, we tested both asymptotic and non-asymptotic log-transformed models to estimate height using diameter and density as predictors. As it turned out, the non-asymptotic models are slightly more precise. To make sure that this result is not an artifact of a specific equation form, we tested models of each form (table 3). For practical use we recommend the most precise model, the last in table 3.

CONCLUSIONS

Diameter and height provide us information about stem taper and ultimately tree volume. Often height is estimated using the easier obtained diameter. However, prediction of height using only diameter does not account for differences in stem taper associated with changes in density for stands of the same diameter. Density helps to explain variation in height and therefore needs to be included into the height-diameter relationship. The relationship between height of trees with the same diameter and density is concave down. Yet, it is not bounded by an asymptote. The model we recommend (table 3) satisfies all the considered requirements. It is also the most precise.

Still, we are not totally happy with our results. We expected that the introduction of density as a predictor would bring the value of parameter *b* closer to its theoretical value of 0.67. We failed in this respect: the excess of parameter b over 0.67 increased from 0.07 to 0.15 (table 3). Further studies need to be conducted to develop a density module that is both efficient in explaining variation in height and provides *b* with a value close to that predicted theoretically.

Table 3—Comparison of the relationships between height, diameter, and density fitted to 379 obs. from the Monticello thinning and pruning study. D = quadratic mean diameter in inches, H = average height in feet (height corresponding to D), SDI = Reineke's stand density index, SDI0 = minimum value of SDI (onset of competition) equal to 34.03, SEE = standard error of the estimate, Adj. R^2 - is the adjusted R-squared value. Variables were log-transformed prior to fitting. The number after $\pm$ represents the single standard error

Equation	a	b	c	SEE	Adj. R^2
		Normal height-diameter relationship			
$H = aD^b$	9.4734	0.7374		0.0981	0.8763
	$\pm$ 0.3443	$\pm$0.0142			
		Height-diameter relationship with an asymptotic density module			
$H = aD^{b*}$	6.4200	0.8196	0.0723	0.0756	0.9266
$(2\text{-}e^{-cSDI/SDI0})$	$\pm$0.2567	$\pm$0.0121	$\pm$0.0074		
		Height-diameter relationship with a non-asymptotic density module			
$H = aD^{b*}$	5.8751	0.8210	0.1945	0.0750	0.9278
$(1\text{+}SDI/SDI0)^c$	$\pm$0.2392	$\pm$0.0120	$\pm$0.0118		
$H = aD^{b*}$	6.5875	0.8223	0.1422	0.0749	0.9280
$(SDI/SDI0)^c$	$\pm$0.2353	$\pm$0.0120	$\pm$0.0086		

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